Full regularity for a C*-algebra of the Canonical Commutation Relations

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Abstract. The Weyl algebra,- the usual C*-algebra employed to model the canonical commutation relations (CCRs), has a well-known defect in that it has a large number of representations which are not regular and these cannot model physical fields. Here, we construct explicitly a C*-algebra which can reproduce the CCRs of a countably dimensional symplectic space (S,B) and such that its representation set is exactly the full set of regular representations of the CCRs. This construction uses Blackadar's version of infinite tensor products of nonunital C*-algebras, and it produces a "host algebra" (i.e. a generalised group algebra, explained below) for the σ–representation theory of the abelian group S where $\sigma(\cdot,\cdot){:=}e^{iB(\cdot,\cdot)/2}$. As an easy application, it then follows that for every regular representation of $\overline{\Delta(S,B)}$ on a separable Hilbert space, there is a direct integral decomposition of it into irreducible regular representations (a known result).
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Introduction

In the description of quantum systems one typically deals with a set of operators satisfying canonical commutation relations. This means that there is a real linear map φ from a given symplectic space $(S,\,B)$ to a linear space of selfadjoint operators on some common dense invariant core $\mathcal{D}$ in a Hilbert space $\mathcal{H}$, satisfying the relations

$$\big[\varphi(f),\,\varphi(g)\big]=iB(f,\,g)\,\mathbb{1},\quad \varphi(f)^*=\varphi(f)\quad\text{on}\quad\mathcal{D}\,.$$

If $\{q_i,p_i\mid i\in I\}\subset S$ is a symplectic basis for S i.e., $0=B(q_i,q_j)=B(p_i,p_j)=B(p_i,q_j)-\delta_{ij}$ then $\varphi(q_i)$ and $\varphi(p_i)$ are interpreted as quantum mechanical position and momentum operators. If S consists of Schwartz functions on a space-time manifold, we can take φ to be a bosonic quantum field.

As is known, if $(S,\,B)$ is non–degenerate then the operators $\varphi(f)$ cannot all be bounded, so it is natural to go from the polynomial algebra $\mathcal{P}$ generated by $\{\varphi(f)\,\big|\,f\in S\}$ to a C*–algebra encoding the same algebraic information. The obvious way to do this, is to form suitable bounded functions of the fields $\varphi(f)$. Following Weyl, we consider the C*–algebra generated by the set of unitaries

$$\big\{\exp\big(i\varphi(f)\big)\,\big|\,f\in S\big\}$$

and this C*–algebra is simple. It can be defined abstractly as the C*–algebra generated by a set of unitaries $\{\delta_f \mid f \in X\}$ subject to the relations $\delta_f^* = \delta_{-f}$ and $\delta_f \delta_g = e^{-iB(f,g)/2}\delta_{f+g}$. This is the familiar Weyl (or CCR) algebra, often denoted $\overline{\Delta(S,\, B)}$ (cf. [M68]). A different C*-algebra for the CCRs was defined in [BG08] based on the resolvents of the fields.

By its definition, $\overline{\Delta(S,\, B)}$ has a representation in which the unitaries δ_f can be identified with the exponentials $e^{i\varphi(f)}$, and hence we can obtain the concrete algebra $\mathcal{P}$ back from these. Such representations $\pi : \overline{\Delta(S,\, B)} \to \mathcal{B}(\mathcal{H})$, i.e. those for which the one–parameter groups $\lambda \to \pi(\delta_{\lambda f})$ are strong operator continuous for all $f \in X$ are called *regular*, and states are regular if their GNS–representations are. Since for physical situations the quantum fields are defined as the generators of the one–parameter groups $\lambda \to \pi(\delta_{\lambda f})$, the representations of interest are required to be regular. (Note that the ray–continuity of $s \to \pi(\delta_s)$ implies continuity on all finite dimensional subspaces of S.) Unfortunately, $\overline{\Delta(S,\, B)}$ has a large number of nonregular representations, and so one can object that it is not satisfactory, since analysis of physical objects can lead to nonphysical ones, e.g. w*-limits of regular states can be nonregular. Nonregular representations are interpreted as situations where the field $\varphi(f)$ can have "infinite field strength". Whilst this is useful for some nonphysical idealizations e.g. plane waves (cf. [AMS93]), or for quantum constraints (cf. [GH88]), for physical situations one wants to exclude such representations. The resolvent algebra of [BG08] also has nonregular representations (although far fewer than the Weyl algebra). Our aim here is to construct a C*-algebra for the CCRs of a countably dimensional (S, B) such that its representation space comprises of exactly the regular representations of the CCRs, in a sense to be made precise below. This will demonstrate that the regular representation theory of the Weyl algebra is isomorphic to the full representation theory of a C*-algebra, and hence it is subject to the usual structure theory for the full representation theory of C*-algebras. The existence of such an algebra has already been shown in [Gr97], but here we want to obtain an explicit construction of it.

In the case that S is finite dimensional, there is an immediate solution. Regard S with its addition as an Abelian group, then $\sigma(\cdot,\cdot) := \exp[iB(\cdot,\cdot)/2]$ is a 2-cocycle of S, and $\overline{\Delta(S,\, B)}$ is just the σ–twisted group algebra of S with the discrete topology (cf. [PR89]). Define the C*–algebra $\mathcal{L}$ as the C*–envelope of the twisted convolution algebra of S, where the latter consists of $L^1(S)$ equipped with the multiplication and involution:

$$f * g(x) = \int_S f(y)\, g(x-y)\, \sigma(y,\, x)\, d\mu(y)\,, \qquad f^*(x) = \overline{f(-x)}$$

where μ is a Haar measure on S, i.e. $\mathcal{L}$ is the σ–twisted group algebra of S. This algebra $\mathcal{L}$ is known to be isomorphic to the compacts $\mathcal{K}(L^2(S))$ (cf. [Se67] and [Bla06] p206). Then we have an embedding of $\overline{\Delta(S,\, B)}$ into the multiplier algebra of $\mathcal{L}$, $\overline{\Delta(S,\, B)} \subset M(\mathcal{L})$, by the action $\delta_x \cdot f(y) = \sigma(x,\, y)\, f(y-x)$. The unique extensions of representations on $\mathcal{L}$ to $\overline{\Delta(S,\, B)}$ produces a bijection from the representations of $\mathcal{L}$ onto the regular representations of $\overline{\Delta(S,\, B)}$

and the bijection respects direct sums and takes irreducibles to irreducibles. So $\mathcal{L}$ is the desired C*-algebra with full regularity.

For the case that S is infinite dimensional, since regular representations π are characterized by requiring the maps $s \to \pi(\delta_s)$ to be continuous on all finite dimensional subspaces of S, this means that we require these maps to be strong operator continuous w.r.t. the inductive limit topology, where the inductive limit is the one consisting of all finite dimensional subspaces of S under inclusion. This inductive limit topology on S is only a group topology w.r.t. addition in the case that S is a countably dimensional space; cf. [Gl03]. Hence in this case the regular representation theory of $\overline{\Delta(S,\, B)}$ is the σ–representation theory of the topological group S, but not otherwise. Henceforth we will always take (S, B) to be countably dimensional, equipped with the (locally convex) inductive limit topology. The problem now becomes the one of how to define a σ–twisted group algebra for S. The usual theory fails in this case, since S is not locally compact, hence there is no Haar measure.

We see that there is a need to generalize the notion of a (twisted) group algebra to topological groups which are not locally compact. Such a generalization, called a *full host algebra,* has been proposed in [Gr05]. Briefly, it is a C^*-algebra $\mathcal{A}$ which has in its multiplier algebra $M(\mathcal{A})$ a homomorphism $\eta: G \to U(M(\mathcal{A}))$, such that the (unique) extension of the representation theory of $\mathcal{A}$ to $M(\mathcal{A})$ pulls back via η to the continuous (unitary) representation theory of G. There is also an analogous concept for unitary σ–representations, where σ is a continuous $\mathbb{T}$-valued 2-cocycle on G. Thus, given a full host algebra $\mathcal{A}$, the continuous representation theory of G can be analyzed on $\mathcal{A}$ with a large arsenal of C^*-algebraic tools.

Our main result in this paper is an explicit construction of a full host algebra for the σ–representations of an infinite dimensional topological linear space S, regarded as a group where S will be a countably dimensional symplectic space with symplectic form B, equipped with the (locally convex) inductive limit topology. We demonstrate the usefulness of this construction by proving that for every regular representation of $\overline{\Delta(S,\, B)}$ on a separable Hilbert space, there is a direct integral decomposition of it into irreducible regular representations. This last result is already known by different means (cf. [He71, Sch90]).

This paper is structured as follows. In Section I we state the notation and definitions necessary for the subsequent material, and in Section II we discuss existence and uniqueness issues for host algebras. In Section III we construct the host algebra for the pair (S, σ) mentioned above, do the direct integral decomposition mentioned, and in the appendix we add general results concerning host algebras and the strict topology which are required for our proofs. These results are of independent interest for the general structure theory of host algebras. The reader in a hurry can skip Section II.

I. Definitions and notation

We will need the following notation and concepts for our main results.

- In the following, we write $M(\mathcal{A})$ for the multiplier algebra of a C^*-algebra $\mathcal{A}$ and, if $\mathcal{A}$ has a unit, $U(\mathcal{A})$ for its unitary group. We have an injective morphism of C^*-algebras $\iota_{\mathcal{A}}\colon \mathcal{A} \to M(\mathcal{A})$ and will just denote $\mathcal{A}$ for its image in $M(\mathcal{A})$. Then $\mathcal{A}$ is dense in $M(\mathcal{A})$ with respect to the *strict topology*, which is the locally convex topology defined by the seminorms

$$p_a(m) := \|m \cdot a\| + \|a \cdot m\|, \qquad a \in \mathcal{A},\ m \in M(\mathcal{A})$$

(cf. [Wo95]).

- For a complex Hilbert space $\mathcal{H}$, we write $\mathrm{Rep}(\mathcal{A}, \mathcal{H})$ for the set of non-degenerate representations of $\mathcal{A}$ on $\mathcal{H}$. Note that the collection $\mathrm{Rep}\,\mathcal{A}$ of all non-degenerate representations of $\mathcal{A}$ is not a set, but a (proper) class in the sense of von Neumann–Bernays–Gödel set theory, cf. [TZ75], and in this framework we can consistently manipulate the object $\mathrm{Rep}\,\mathcal{A}$. However, to avoid set–theoretical subtleties, we will express our results below concretely, i.e., in terms of $\mathrm{Rep}(\mathcal{A}, \mathcal{H})$ for given Hilbert spaces $\mathcal{H}$. We have an injection

$$\mathrm{Rep}(\mathcal{A}, \mathcal{H}) \hookrightarrow \mathrm{Rep}(M(\mathcal{A}), \mathcal{H}), \quad \pi \mapsto \widetilde{\pi} \quad \text{with} \quad \widetilde{\pi} \circ \iota_{\mathcal{A}} = \pi,$$

which identifies the non-degenerate representation π of $\mathcal{A}$ with that representation $\widetilde{\pi}$ of its multiplier algebra which extends π and is continuous with respect to the strict topology on $M(\mathcal{A})$ and the topology of pointwise convergence on $B(\mathcal{H})$.

- For topological groups G and H we write $\mathrm{Hom}(G, H)$ for the set of continuous group homomorphisms $G \to H$. We also write $\mathrm{Rep}(G, \mathcal{H})$ for the set of all (strong operator) continuous unitary representations of G on $\mathcal{H}$. Endowing $U(\mathcal{H})$ with the strong operator topology turns it into a topological group, denoted $U(\mathcal{H})_s$, so that $\mathrm{Rep}(G, \mathcal{H}) = \mathrm{Hom}(G, U(\mathcal{H})_s)$.

- Let $\mathbb{T} \subseteq \mathbb{C}^\times$ denote the unit circle, viewed as a multiplicative subgroup and $\sigma\colon G \times G \to \mathbb{T}$ be a continuous 2-cocycle, i.e.,

$$\sigma(\mathbf{1}, x) = \sigma(x, \mathbf{1}) = 1, \quad \sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z) \quad \text{for} \quad x, y, z \in G.$$

We then form the topological group

$$G_\sigma := \mathbb{T} \times G, \quad (t, g)(t', g') := (tt'\sigma(g, g'), gg')$$

and note that the projection $q\colon G_\sigma \to G$ defines a central extension of G by $\mathbb{T}$. A continuous unitary representation $(\pi, \mathcal{H})$ of G_σ is called a *σ-representation of G* if $\pi(t, \mathbf{1}) = t\mathbf{1}$ holds for each $t \in \mathbb{T}$. Then

$$G \to U(\mathcal{H}), \quad g \mapsto \pi(1, g)$$

is continuous with respect to the strong operator topology, but

$$\pi(1,g)\pi(1,g') = \sigma(g,g')\pi(1,gg') \quad \text{for} \quad g,g' \in G.$$

We write $\mathrm{Rep}((G,\sigma),\mathcal{H})$ for the set of all continuous σ-representations of G on $\mathcal{H}$.

Definition I.1. Let G be a topological group and $\sigma: G \times G \to \mathbb{T}$ a continuous 2-cocycle. A *host algebra for the pair* (G,σ) is a pair $(\mathcal{L},\eta)$ where $\mathcal{L}$ is a C^*-algebra and $\eta: G_\sigma \to U(M(\mathcal{L}))$ is a homomorphism such that for each complex Hilbert space $\mathcal{H}$ the corresponding map

$$\eta^*: \mathrm{Rep}(\mathcal{L},\mathcal{H}) \to \mathrm{Rep}((G,\sigma),\mathcal{H}), \quad \pi \mapsto \widetilde{\pi} \circ \eta$$

is injective. We then write $\mathrm{Rep}(G,\mathcal{H})_\eta \subseteq \mathrm{Rep}(G,\mathcal{H})$ for the range of η^*. We say that (G,σ) has a *full host algebra* if it has a host algebra for which η^* is surjective for each Hilbert space $\mathcal{H}$.

In the case that $\sigma = 1$, we simply speak of a *host algebra for* G. In this case, $G_\sigma = G \times \mathbb{T}$ is a direct product, so that a host algebra for G is a pair $(\mathcal{L},\eta)$, where $\eta: G \to U(M(\mathcal{L}))$ is a homomorphism into the unitary group of $M(\mathcal{L})$ such that for each complex Hilbert space $\mathcal{H}$ the corresponding map

$$\eta^*: \mathrm{Rep}(\mathcal{L},\mathcal{H}) \to \mathrm{Rep}(G,\mathcal{H}), \quad \pi \mapsto \widetilde{\pi} \circ \eta$$

is injective. We then write $\mathrm{Rep}(G,\mathcal{H})_\eta \subseteq \mathrm{Rep}(G,\mathcal{H})$ for the range of η^*. We say that G has a *full host algebra* if it has a host algebra for which η^* is surjective for each Hilbert space $\mathcal{H}$.

Note that by the universal property of (twisted) group algebras, the homomorphism $\eta: G_\sigma \to U(M(\mathcal{L}))$ extends uniquely to the σ–twisted group algebra of G with the discrete topology, i.e., we have a *-homomorphism $\eta: C^*_\sigma(G_d) \to U(M(\mathcal{L}))$ (still denoted by η). ■

Remark: (1) It is well known that for each locally compact group G, the group C^*-algebra $C^*(G)$, and the natural map $\eta_G: G \to M(C^*(G))$ provide a full host algebra ([Dix64, Sect. 13.9]) and for each pair (G,σ), where G is locally compact, the corresponding twisted group C^*-algebra $C^*(G,\sigma)$, which is isomorphic to an ideal of $C^*(G_\sigma)$, is a full host algebra for the pair (G,σ). This is most easily seen by decomposition of representations of G_σ into isotypic summands with respect to the action of the central subgroup $\mathbb{T} \times \{\mathbf{1}\}$ (apply [BS70], [PR89] with $\mathcal{L} = \mathbb{C}$). The map $\eta_G : G \to M(C^*(G,\sigma))$ is continuous w.r.t. the strict topology of $M(C^*(G,\sigma))$.[1]

(2) Note that the map η^* preserves direct sums, unitary conjugation, subrepresentations, and for full host algebras, irreducibility (cf. [Gr05]) so that this notion of isomorphism between $\mathrm{Rep}((G,\sigma),\mathcal{H})$ and $\mathrm{Rep}(\mathcal{L},\mathcal{H})$ involves strong structural correspondences.

(3) Whilst the concept of a host algebra is a natural extension of the concept of a group C*-algebra (and it easily generalizes to other algebraic objects cf. [Gr05]), it has so far

[1] This is an easy consequence of the fact that $\mathrm{im}(\eta_G)$ is bounded and that the action on the corresponding L^1-algebra is continuous.

had a troubled history. It was first used in [Gr97], though not under this name. There, the existence of host algebras was proven for groups which are inductive limits of locally compact groups, though the proof was not constructive enough to allow much further structural analysis of these host algebras. Then in [Gr05] the concept was generalised to algebraic objects other than topological groups, and a general existence and uniqueness theorem was given, though unfortunately this turned out to be wrong (see the erratum, and the counterexample below). Since then, host algebras have been constructed in [Ne08] for complex semigroups. Our aim in Section III is to provide an explicit, and more useful construction of a host algebra (than [Gr97]) for the regular representations of the canonical commutation relations.

II. Existence and Uniqueness issues.

For general topological groups, there are serious existence and uniqueness questions for their host algebras (as mentioned, the existence and uniqueness theorem in [Gr05] is wrong). From the structural "isomorphism" between the σ-representation theory of G and the representation theory of its full host noted above, it becomes easy to find examples of topological groups without full host algebras. For instance in Example 5.2 of [Pe97] is an abelian topological group with a faithful continuous unitary representation, but no continuous irreducible representations. Hence this group cannot have a host algebra, whether full or not. In [GN01] it is shown in particular for any non-atomic measure space (X,μ), such as the unit interval $[0,1]$ with Lebesgue meaure, the unitary group of the W^*-algebra $L^\infty(X,\mu)$, endowed with the weak topology, has no non-trivial continuous characters, hence no non-zero host algebra.

It is therefore an important open problem to characterize those pairs (G,σ) for which full host algebras exist.

Concerning the issue of uniqueness, the following simple counterexample shows that if a host algebra exists, then it need not be unique. Let $G := \mathbb{Z}$. Then its character group is $\widehat{G} \cong \mathbb{T}$, which is a compact group with respect to the topology of pointwise convergence. Since G is locally compact, $C^*(G) \cong C(\mathbb{T})$ is a full host algebra for G. Let $\mathcal{L} := C_0([0,1))$ and define a homomorphism $\eta : \mathbb{Z} \to U(M(C_0([0,1)))) \cong C([0,1),\mathbb{T})$ by $\eta(n)(x) := e^{2\pi inx}$. Then $\eta(1) : [0,1) \to \mathbb{T}$ is a continuous bijection, which implies that $\eta(\mathbb{Z})$ separates the points, hence by Lemma A.1 below the C^*-algebra generated by this set is strictly dense in $M(\mathcal{L})$. Since the unique extensions of representations of $\mathcal{L}$ to $M(\mathcal{L})$ are continuous in the strict topology, it follows that η^* is injective. Further, $\mathbb{Z}$ is discrete, so that continuity of representations $\eta^*\pi$ is trivially satisfied, and thus $(\mathcal{L},\eta)$ is a host algebra. This host algebra is full because the representations of $\mathbb{Z}$ are in one-to-one correspondence with Borel spectral measures on $\mathbb{T}$ and $\eta(1)$ is a Borel isomorphism. Note in particular that this full host algebra $\mathcal{L} \ncong C^*(G)$ is not unital, although G is a discrete group.

This issue also needs further analysis, e.g. one needs to find what structural properties are shared by host algebras for the same pair (G,σ), and to explore the properties of the set of host algebras. In the appendix we list more host algebra properties, e.g. those relating to products and homomorphisms of groups.

III. A construction of a full host algebra for $(S,\,\sigma)$.

Here we want to present an example of a host algebra for an infinite-dimensional group. Let (S,B) be a countably dimensional (nondegenerate) symplectic space. Then by Lemma A.8 we know that there is a complex structure and a hermitian inner product $(\cdot,\cdot)$ on S such that $B(v,\,w)=\mathrm{Im}\,(v,\,w)$ for all $v,\,w\in S$. Moreover, w.r.t. the inner product $(\cdot,\cdot)$, S has an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. We consider $S\cong\mathbb{C}^{(\mathbb{N})}$ as an inductive limit of the subspaces $S_n:=\mathrm{span}\{e_1,\ldots,e_n\}$ and endow it with the inductive limit topology, which turns it into an abelian topological group with respect to addition (which is only true for countably dimensional spaces; cf. [Gl03]). Moreover, the symplectic form $B(v,\,w)=\mathrm{Im}\,(v,\,w)$ defines a group two-cocycle $\sigma(v,\,w):=\exp[iB(v,w)/2]$ on S. Let S_σ denote the corresponding central extension of S by $\mathbb{T}$ (cf. above Definition I.1). In the rest of this section we will prove that:

Theorem III.1. *The pair (S,σ) has a full host algebra.* ∎

Recall that $\mathcal{A}:=\overline{\Delta(S,\,B)}$ is the discrete twisted σ–group algebra of S, i.e., it is the unique (simple) C^*-algebra generated by a collection of unitaries $\big\{\delta_s\;\big|\;s\in S\big\}$ satisfying the (Weyl) relations $\delta_{s_1}\delta_{s_2}=\sigma(s_1,s_2)\,\delta_{s_1+s_2}$ ([BR97, Th. 5.2.8]). Let

$$\mathcal{R}(\mathcal{H}):=\big\{\pi\in\mathrm{Rep}(\mathcal{A},\mathcal{H})\;\mid\;t\in\mathbb{R}\to\pi(\delta_{tx})\text{ is strong operator continuous }\forall x\in S\big\}$$

denote the set of regular representations on the Hilbert space $\mathcal{H}$. Through the identification $\pi(s):=\pi(\delta_s)$, $\mathcal{R}(\mathcal{H})$ corresponds exactly with the σ-representations of S on $\mathcal{H}$, i.e., with $\mathrm{Rep}((S,\sigma),\mathcal{H})$.

Lemma III.2. *With the notation above, we have $\mathcal{A}=\bigotimes\limits_{n=1}^{\infty}\mathcal{A}_n$ with the spatial (minimal) tensor norms, where $\mathcal{A}_n:=\mathrm{C}^*\big\{\delta_{ze_n}\,\big|\,z\in\mathbb{C}\big\}$.*

Proof. This follows directly from Proposition 11.4.3 of Kadison and Ringrose [KR83], we only need to verify that its conditions hold in the present context. For this, observe that $\mathcal{A}=\mathrm{C}^*\{\bigcup\limits_{n=1}^{\infty}\mathcal{A}_n\}$, $\mathbb{1}\in\mathcal{A}_n$, $\big[\mathcal{A}_n,\,\mathcal{A}_m\big]=\{0\}$ when $n\neq m$. Moreover, the linear maps

$$\psi_k:\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_k\to\mathcal{A}\qquad\text{defined by}\qquad\psi_k\big(A_1\otimes\cdots\otimes A_k\big):=A_1A_2\cdots A_k$$

are $*$–monomorphisms because each image subalgebra $\mathrm{C}^*\big\{\bigcup_{n=1}^{k}\mathcal{A}_n\big\}$ is the unique C^*-algebra generated by the unitaries $\big\{\delta_{ze_i}\;\big|\;z\in\mathbb{C},\;i=1,\ldots,k\big\}$, and this is also true for $\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_k$. This is enough to apply the proposition loc. cit. ■

Observe that each $\mathcal{A}_n$ is just the discrete σ-group algebra of the subgroup $\mathbb{C}e_n\subset S$, and as the latter is locally compact, we can construct its σ–twisted group algebra which we denote by $\mathcal{L}_n$ (recall that $\mathcal{L}_n$ is just the enveloping C^*-algebra of $L^1(\mathbb{C})$, equipped with σ-twisted convolution). It is well-known that $\mathcal{L}_n\cong\mathcal{K}(L^2(\mathbb{R}))$ (cf. Segal [Se67]). Note that for each finite subset $F\subset\mathbb{N}$, the algebra $\bigotimes_{n\in F}\mathcal{L}_n\cong\mathcal{K}(\bigotimes_{n\in F}L^2(\mathbb{R}))\cong\mathcal{K}(L^2(\mathbb{R}^F))$ is a host algebra for the regular representations of $\bigotimes_{n\in F}\mathcal{A}_n=\mathrm{C}^*\big\{\delta_{ze_n}\;\big|\;z\in\mathbb{C},\;n\in F\big\}$, i.e., for the σ–representations of $\mathrm{span}\{e_n\,|\,n\in F\}\subset S$.

It is natural to try some infinite tensor product $\bigotimes_{n=1}^{\infty}\mathcal{L}_n$ for a host algebra, but because the algebras $\mathcal{L}_n$ are non-unital, the definition of the infinite tensor product needs some care [Bla77]. For each $n\in\mathbb{N}$, choose a nonzero projection $P_n\in\mathcal{L}_n\cong\mathcal{K}(\mathcal{H})$ and define C^*-embeddings

$$\Psi_{\ell k}:\mathcal{L}^{(k)}\to\mathcal{L}^{(\ell)}\qquad\text{by}\qquad\Psi_{\ell k}(A_1\otimes\cdots\otimes A_k):=A_1\otimes\cdots\otimes A_k\otimes P_{k+1}\otimes\cdots\otimes P_\ell,$$

where $k<\ell$ and $\mathcal{L}^{(k)}:=\bigotimes_{n=1}^{k}\mathcal{L}_n$. Then the inductive limit makes sense, so we define

$$\mathcal{L}:=\bigotimes_{n=1}^{\infty}\mathcal{L}_n:=\varinjlim\big\{\mathcal{L}^{(n)},\,\Psi_{\ell k}\big\}$$

and write $\Psi_k:\mathcal{L}^{(k)}\to\mathcal{L}$ for the corresponding embeddings, satisfying $\Psi_k\circ\Psi_{kj}=\Psi_j$ for $j\le k$. Since each $\mathcal{L}_n$ is simple, so are the finite tensor products $\mathcal{L}^{(k)}$ ([WO93], Prop. T.6.25), and as inductive limits of simple C^*-algebras are simple ([KR83], Prop. 11.4.2), so is $\mathcal{L}$. It is also clear that $\mathcal{L}$ is separable.

Since $\Psi_{k+n,k}(L_k)=L_k\otimes P_{k+1}\otimes\cdots\otimes P_{k+n}$, where $L_k\in\mathcal{L}^{(k)}$, this means that we can consider $\mathcal{L}$ to be built up out of elementary tensors of the form

$$\Psi_k(L_1\otimes\cdots\otimes L_k)=L_1\otimes L_2\otimes\cdots\otimes L_k\otimes P_{k+1}\otimes P_{k+2}\otimes\cdots\;,\qquad\text{where}\qquad L_i\in\mathcal{L}_i\;,\quad(4.1)$$

i.e., eventually they are of the form $\cdots\otimes P_k\otimes P_{k+1}\otimes\cdots$. We will use this picture below, and generally will not indicate the maps Ψ_k.

Lemma III.3.

(i) *With respect to componentwise multiplication, we have an inclusion*

$$\mathcal{A}=\bigotimes_{n=1}^{\infty}\mathcal{A}_n\subset M(\mathcal{L})=M\big(\bigotimes_{n=1}^{\infty}\mathcal{L}_n\big)\;.$$

(ii) *There is a natural embedding* $\iota_n : M(\mathcal{L}^{(n)}) \hookrightarrow M(\mathcal{L})$. *This is a topological embedding on each bounded subset of* $M(\mathcal{L}^{(n)})$. *Moreover,* $\mathcal{L}^{(n)}$ *is dense in* $M(\mathcal{L}^{(n)})$ *with respect to the restriction of the strict topology of* $M(\mathcal{L})$.

(iii) *Let* $\pi \in \operatorname{Rep}(\mathcal{L}, \mathcal{H})$, *and let* π_n *denote the unique representation which it induces on* $\mathcal{L}^{(n)} \subset M(\mathcal{L}^{(n)}) \subset M(\mathcal{L})$ *by strict extension. Then*

$$\pi(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s-lim}_{n\to\infty} \pi_n(L_1 \otimes \cdots \otimes L_n)$$

for all $L_1 \otimes L_2 \otimes \cdots \in \mathcal{L}$ *as in* (4.1).

Proof. (i) For each k we obtain a homomorphism $\Theta_k : \bigotimes_{n=1}^{k} \mathcal{A}_n \to M(\mathcal{L})$ by componentwise multiplication in the first k entries of $\mathcal{L}$, leaving all entries further up invariant. By simplicity of its domain, each Θ_k is a monomorphism. From $\Theta_k\big(\bigotimes_{n=1}^{k} \mathcal{A}_n\big) \subset M(\mathcal{L})$ for each $k \in \mathbb{N}$, we obtain all the generating unitaries δ_s in $M(\mathcal{L})$, then they generate $\mathcal{A}$ in $M(\mathcal{L})$ by uniqueness of the C^*-algebra of the canonical commutation relations.

(ii) Now $\mathcal{L} = \mathcal{L}^{(n)} \otimes \mathcal{B}$ for a C^*-algebra $\mathcal{B}$ (cf. Blackadar [Bla77, p. 315]), and $M(\mathcal{L}^{(n)})$ embeds in $M(\mathcal{L})$ as $M(\mathcal{L}^{(n)}) \otimes \mathbb{1}$. Therefore (ii) follows from Lemma A.2.

(iii) Note that $U_n := \Psi_n(\mathbb{1}) = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes P_{n+1} \otimes P_{n+2} \otimes \cdots \in M(\mathcal{L})$ converges strictly to $\mathbb{1}$. Recall that $L = L_1 \otimes L_2 \otimes \cdots \in \mathcal{L}$ as in (4.1) is of the form

$$A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes P_{k+1} \otimes P_{k+2} \otimes \cdots ,$$

where $A_i \in \mathcal{L}_i$, so for $n \geq k$ we get for all $\psi \in \mathcal{H}_\pi$ that for the strictly continuous extension $\widetilde{\pi}$ of π to $M(\mathcal{L})$:

$$\begin{aligned}
\big\|\widetilde{\pi}(L - L_1 \otimes \cdots \otimes L_n \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots)\psi\big\| &= \Big\|\widetilde{\pi}\big(L_1 \otimes \cdots \otimes L_n \otimes (P_{n+1} \otimes P_{n+2} \otimes \cdots - \mathbb{1})\big)\psi\Big\| \\
&= \Big\|\widetilde{\pi}\big(L_1 \otimes \cdots \otimes L_n \otimes \mathbb{1} \otimes \cdots\big) \cdot \widetilde{\pi}(U_n - \mathbb{1})\psi\Big\| \\
&\leq C \cdot \big\|\pi(U_n - \mathbb{1})\psi\big\| \to 0
\end{aligned}$$

as $n \to \infty$, where $C > 0$ is chosen such that $\|L_1 \otimes \cdots \otimes L_n\| \leq C$ for all n, and this is possible because $\|P_{k+1} \otimes P_{k+2} \otimes \cdots\| = 1$. But this is exactly the claim we needed to prove. ■

Let $\pi \in \operatorname{Rep}(\mathcal{A}, \mathcal{H})$ be regular. Observe that π is regular on all $\mathcal{A}_n$, hence there are unique $\widehat{\pi}_n \in \operatorname{Rep}(\mathcal{L}_n, \mathcal{H})$ which extend (on $\mathcal{H}$) to $\pi\restriction\mathcal{A}_n$ by the host algebra property of $\mathcal{L}_n$. For the distinguished projections $P_n \in \mathcal{L}_n$, we simplify the notation to $\pi(P_n) := \widehat{\pi}_n(P_n)$. Observe that the projections $\pi(P_j)$ all commute, and so the strong limit

$$\mathbb{P}_k := \operatorname*{s-lim}_{n\to\infty} \pi(P_k) \cdots \pi(P_n)$$

exists, and it is the projection onto the intersection of the ranges of of all $\pi(P_j)$, $j \geq k$. Since $\mathbb{P}_k = \pi(P_k)\,\mathbb{P}_{k+1}$ we have $\mathbb{P}_{k+1} \geq \mathbb{P}_k$ and so also $\operatorname*{s-lim}_{k\to\infty} \mathbb{P}_k \leq \mathbb{1}$ exists.

We will use the notation $\mathcal{A}^{(n)} := \bigotimes_{j=1}^{n} \mathcal{A}_j$ below.

Proposition III.4. *Define a monomorphism $\eta : S_\sigma \to U(M(\mathcal{L}))$ by $\eta((s,t)) := t\delta_s \in \mathcal{A} \subset M(\mathcal{L})$ (by Lemma III.3(i)). Then η is continuous with respect to the strict topology on $M(\mathcal{L})$ and $\mathcal{L}$ is a host algebra of (S,σ), i.e., the maps $\eta^* : \mathrm{Rep}(\mathcal{L},\mathcal{H}) \to \mathrm{Rep}((S,\sigma),\mathcal{H})$ are injective. The range of η^* consists of those $\pi \in \mathrm{Rep}\big((S,\sigma),\mathcal{H}\big)$ for which $\operatorname*{s-lim}_{k\to\infty} \mathbb{P}_k = \mathbb{1}$.*

Proof. Let π be a representation of $\mathcal{L}$ and $\widetilde{\pi}$ its strictly continuous extension to $M(\mathcal{L})$. To see that the representation $\eta^*\widetilde{\pi}$ of S_σ is continuous, we show that η is continuous with respect to the strict topology on $M(\mathcal{L})$. Since S_σ is a topological direct limit of the subgroups $S_{m,\sigma}$, where $S_m = \mathrm{span}_{\mathbb{C}}\{e_1,\ldots,e_m\}$, it suffices to show that η is continuous on each subgroup $S_{m,\sigma}$. Recall that the twisted group algebra $C^*(S_m,\sigma) \cong \mathcal{L}^{(m)}$ is a full host algebra for (S_m,σ) and that the corresponding strictly continuous homomorphism $\eta_m : S_{m,\sigma} \to M(\mathcal{L}^{(m)})$ is compatible with the embedding $\iota_m : M(\mathcal{L}^{(m)}) \hookrightarrow M(\mathcal{L})$ in the sense that $\eta|_{S_{m,\sigma}} = \iota_m \circ \eta_m$. Since ι_m restricts to an embedding on the unitary group (Lemma III.3(ii)), the continuity of η_m implies the continuity of η on $S_{m,\sigma}$, which in turn implies the continuity of η. As a consequence, $\widetilde{\pi} \circ \eta$ is a continuous unitary representation of S_σ for each strictly continuous representation $\widetilde{\pi}$ of $M(\mathcal{L})$.

To see that η^* is injective, we have to show that two representations π_1, π_2 of $\mathcal{L}$ for which $\eta^*\pi_1 = \eta^*\pi_2$ are equal. If $\eta^*\pi_1 = \eta^*\pi_2$, then we obtain for each $m \in \mathbb{N}$ the relation $\eta_m^*\pi_1 = \eta_m^*\pi_2$ on $S_{m,\sigma}$. This means that the corresponding unitary representations of the group $S_{m,\sigma}$ coincide. In view of Lemma III.3(iii), it suffices to argue that the two non-degenerate representations $\pi_{1,m}$ and $\pi_{2,m}$ of $\mathcal{L}^{(m)}$ coincide (cf. Lemma A.3 for the non-degeneracy), which in turn follows from the host algebra property of $\mathcal{L}^{(m)}$ for $S_{m,\sigma}$.

To characterize the range of η^*, let $\pi \in \mathrm{Rep}(\mathcal{A},\mathcal{H})$ be the strictly continuous extension of a $\pi_0 \in \mathrm{Rep}(\mathcal{L},\mathcal{H})$. Then, by Lemma III.3(iii), it must satisfy

$$\pi_0(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s-lim}_{n\to\infty} \pi_n(L_1 \otimes \cdots \otimes L_n)$$

for all $L_1 \otimes L_2 \otimes \cdots \in \mathcal{L}$. Now we have

$$\pi_n(L_1 \otimes \cdots \otimes L_{n-1} \otimes P_n) = \widetilde{\pi}_n(L_1 \otimes \cdots \otimes L_{n-1} \otimes \mathbb{1})\, \widetilde{\pi}_n(\mathbb{1} \otimes \cdots \mathbb{1} \otimes P_n)$$

where $\widetilde{\pi}_n$ denotes the strictly continuous extension to $M(\mathcal{L}^{(n)})$, and it is obvious that these two operators commute. From the algebra relations $\mathcal{A}^{(n)} \supset \mathcal{A}^{(n-1)} \subset M(\mathcal{L}^{(n-1)}) \subset M(\mathcal{L}^{(n)})$, and the host algebra properties we get that $\widetilde{\pi}_n(L_1 \otimes \cdots \otimes L_{n-1} \otimes \mathbb{1}) = \pi_{n-1}(L_1 \otimes \cdots \otimes L_{n-1})$ and $\widetilde{\pi}_n(\mathbb{1} \otimes \cdots \mathbb{1} \otimes P_n) = \pi(P_n)$, so

$$\pi_n(L_1 \otimes \cdots \otimes L_{n-1} \otimes P_n) = \pi_{n-1}(L_1 \otimes \cdots \otimes L_{n-1})\, \pi(P_n)\ .$$

Thus, for

$$L = L_1 \otimes L_2 \otimes \cdots = A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes P_{k+1} \otimes P_{k+2} \otimes \cdots \in \mathcal{L}, \quad \text{we get for } n > k :$$

$$\pi_n(L_1 \otimes \cdots \otimes L_n) = \pi_k(A_1 \otimes \cdots \otimes A_k)\, \pi(P_{k+1}) \cdots \pi(P_n)\ .$$

Using the fact that the projections $\pi(P_j)$ all commute,

$$\pi_0(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s-lim}_{n\to\infty} \pi_n(L_1 \otimes \cdots \otimes L_n) = \pi_k(A_1 \otimes \cdots \otimes A_k)\,\mathbb{P}_{k+1}\,.$$

Since π_0 is non-degenerate, and all $\pi_k \restriction \mathcal{L}^{(k)}$ are non-degenerate, it follows that $\operatorname*{s-lim}_{k\to\infty} \mathbb{P}_k = \mathbb{1}$.

Conversely, if we start from a regular representation π of $\mathcal{A}$ which satisfies $\operatorname*{s-lim}_{k\to\infty} \mathbb{P}_k = \mathbb{1}$, we will *define* a representation π_0 on $\mathcal{L}$ by

$$\pi_0(L) := \pi_k(A_1 \otimes \cdots \otimes A_k)\,\mathbb{P}_{k+1} \quad \text{for} \quad L = A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes P_{k+1} \otimes P_{k+2} \otimes \cdots$$

where $\pi_k \in \operatorname{Rep}\mathcal{L}^{(k)}$ is obtained from $\pi \restriction \mathcal{A}^{(k)}$, using the host algebra property of $\mathcal{L}^{(k)}$. To see that this can be done, note that for $A \in \mathcal{L}^{(k)}$ we have

$$\pi_k(A)\mathbb{P}_{k+1} = \pi_{k+1}(\Psi_{k+1,k}(A))\mathbb{P}_{k+2}.$$

Therefore the universal property of the direct limit algebra $\mathcal{L}$ implies the existence of a representation π_0 of $\mathcal{L}$, satisfying

$$\pi_0(\Psi_k(A)) = \pi_k(A)\mathbb{P}_{k+1} \quad \text{for} \quad A \in \mathcal{L}^{(k)}.$$

That it is non-degenerate follows from the fact that each π_k is non-degenerate, and that $\operatorname*{s-lim}_{k\to\infty} \mathbb{P}_k = \mathbb{1}$. To see that $\widetilde{\pi}_0 \restriction \mathcal{A} = \pi$, recall that π_k is the representation obtained from from $\pi \restriction \mathcal{A}^{(k)}$, using the host algebra property of $\mathcal{L}^{(k)}$. Let $B \in \mathcal{A}^{(k)}$, then for $A \in \mathcal{L}^{(k)}$ we have

$$\widetilde{\pi}_0(B)\,\pi_0(\Psi_k(A)) = \pi_0(B \cdot \Psi_k(A)) = \pi_k(B \cdot A)\mathbb{P}_{k+1} = \pi(B)\,\pi_k(A)\mathbb{P}_{k+1} = \pi(B)\,\pi_0(\Psi_k(A))$$

from which it follows that $\widetilde{\pi}_0 \restriction \mathcal{A} = \pi$. ■

Thus for every family of projections $P_k \in \mathcal{L}_k$ we get a host algebra. Now recall that $\mathcal{L}_k \cong \mathcal{K}(\ell^2(\mathbb{N}))$, and that there is a (countable) approximate identity $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(\ell^2(\mathbb{N}))$ consisting of a strictly increasing sequence of projections E_n with $\dim(E_n\ell^2(\mathbb{N})) = n$. For each k, choose such an approximate identity $(E_n^{(k)}) \subset \mathcal{L}_k$, then for each sequence $\mathbf{n} = (n_1, n_2, \ldots) \in \mathbb{N}^\infty := \mathbb{N}^{\mathbb{N}}$, we have a sequence of projections $\big(E_{n_1}^{(1)}, E_{n_2}^{(2)}, \ldots\big)$ from which we can construct an infinite tensor product as above, and we will denote it by $\mathcal{L}[\mathbf{n}]$. For the elementary tensors, we streamline the notation to:

$$A_1 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1} := A_1 \otimes \cdots \otimes A_k \otimes E_{n_{k+1}}^{(k+1)} \otimes E_{n_{k+2}}^{(k+2)} \otimes \cdots \in \mathcal{L}[\mathbf{n}]$$

where $A_i \in \mathcal{L}_i$, and their closed span is the simple C^*-algebra $\mathcal{L}[\mathbf{n}]$.

Next we want to define componentwise multiplication between different C*-algebras $\mathcal{L}[\mathbf{n}]$ and $\mathcal{L}[\mathbf{m}]$. This can of course be done in the algebraic infinite tensor product of the algebras

$\mathcal{L}_k$, (cf. [Bo74, p470]) using suitable closures of subalgebras, but it is faster to proceed as follows. Note that for componentwise multiplication, the sequences give:

$$\big(E^{(1)}_{n_1},\,E^{(2)}_{n_2},\ldots\big)\cdot\big(E^{(1)}_{m_1},\,E^{(2)}_{m_2},\ldots\big)=\big(E^{(1)}_{p_1},\,E^{(2)}_{p_2},\ldots\big)$$

where $p_j:=\min(n_j,\,m_j)$, i.e., multiplication reduces the entries, and hence the sequence $\big(E^{(1)}_1,\,E^{(2)}_1,\,E^{(3)}_1\ldots\big)$ is invariant under such multiplication. So we define an embedding $\mathcal{L}[\mathbf{n}]\subseteq M(\mathcal{L}[\mathbf{1}])$ for all $\mathbf{n}$, where $\mathbf{1}:=(1,\,1,\ldots)$ by

$$\begin{aligned}&\big(A_1\otimes\cdots\otimes A_k\otimes E[\mathbf{n}]_{k+1}\big)\cdot\big(B_1\otimes\cdots\otimes B_n\otimes E[\mathbf{1}]_{n+1}\big)\\&:=\begin{cases}A_1B_1\otimes\cdots\otimes A_nB_n\otimes A_{n+1}E^{(n+1)}_1\cdots\otimes A_kE^{(k)}_1\otimes E[\mathbf{1}]_{k+1} & \text{if } n\leq k\\ A_1B_1\otimes\cdots\otimes A_kB_k\otimes E^{(k+1)}_{n_{k+1}}B_{k+1}\cdots\otimes E^{(n)}_{n_n}B_n\otimes E[\mathbf{1}]_{n+1} & \text{if } n\geq k\end{cases}\end{aligned}$$

for the left action, and similar for the right action on $\mathcal{L}[\mathbf{1}]$. To see that this is an embedding as claimed, choose a faithful representation π_i of each $\mathcal{L}_i\cong\mathcal{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}_i$ and let ψ_n be a unit vector in $E^{(n)}_1\mathcal{H}_n$. Construct the infinite tensor product Hilbert space $\bigotimes\limits_{n=1}^{\infty}\mathcal{H}_n$ w.r.t. the sequence $(\psi_1,\,\psi_2,\ldots)$, and note that for each $\mathcal{L}[\mathbf{n}]$, the tensor representation $\bigotimes\limits_{n=1}^{\infty}\pi_n$ on $\bigotimes\limits_{n=1}^{\infty}\mathcal{H}_n$ is faithful (since it is faithful on the C*-algebras of which they are inductive limits). Then it is obvious that the given multiplication above is concretely realised on this Hilbert space, and by faithfulness of the representations we realise the embeddings $\mathcal{L}[\mathbf{n}]\subseteq M(\mathcal{L}[\mathbf{1}])$ for all $\mathbf{n}$. Then

$$\mathcal{L}[\mathbf{n}]\cdot\mathcal{L}[\mathbf{m}]\subseteq\mathcal{L}[\mathbf{p}], \tag{4.3}$$

where $p_j:=\min(n_j,\,m_j)$, and in fact

$$\mathcal{L}[\mathbf{n}]\subset M(\mathcal{L}[\mathbf{p}])\supset\mathcal{L}[\mathbf{m}]. \tag{4.4}$$

Using the embedding $\mathcal{L}[\mathbf{n}]\subseteq M(\mathcal{L}[\mathbf{1}])$ for all $\mathbf{n}$, we define the C^*-algebra in $M(\mathcal{L}[\mathbf{1}])$ generated by all $\mathcal{L}[\mathbf{n}]$, and denote it by $\mathcal{L}[E]$. By (4.3), this is just the closed span of all $\mathcal{L}[\mathbf{n}]$ and hence the closure of the dense *-subalgebra $\mathcal{L}_0\subset\mathcal{L}[E]$, where

$$\mathcal{L}_0:=\sum_{\mathbf{n}\in\mathbb{N}^\infty}\mathcal{L}[\mathbf{n}]_0\quad\text{(finite sums) and}\quad\mathcal{L}[\mathbf{n}]_0:=\bigcup_{k\in\mathbb{N}}\mathcal{L}^{(k)}\otimes E[\mathbf{n}]_{k+1}.$$

We still have $\mathcal{A}\subset M(\mathcal{L}[E])\supset\mathcal{L}^{(n)}$ for each $n\in\mathbb{N}$. Note that if two sequences $\mathbf{n}$ and $\mathbf{m}$ differ only in a finite number of entries, then $\mathcal{L}[\mathbf{n}]=\mathcal{L}[\mathbf{m}]$, and hence we actually have that the correct index set for the algebras $\mathcal{L}[\mathbf{n}]$ is not the sequences $\mathbb{N}^\infty$, but the set of equivalence classes $\mathbb{N}^\infty/\!\sim$ where $\mathbf{n}\sim\mathbf{m}$ if they differ only in finitely many entries. Some of the structures of $\mathbb{N}^\infty$ will factor through to $\mathbb{N}^\infty/\!\sim$, e.g. we have a partial ordering of equivalence classes defined

by $[\mathbf{n}] \geq [\mathbf{m}]$ if for any representatives $\mathbf{n}$ and $\mathbf{m}$ resp., we have that there is an N (depending on the representatives) such that $n_k \geq m_k$ for all $k > N$. In particular, we note that products reduce sequences, i.e., we have $\mathcal{L}[\mathbf{n}] \cdot \mathcal{L}[\mathbf{p}] \subseteq \mathcal{L}[\mathbf{q}]$ for $q_i = \min(n_i, p_i)$, so $[\mathbf{n}] \geq [\mathbf{q}] \leq [\mathbf{p}]$.

Let $\varphi : \mathbb{N}^\infty/\!\sim\, \to \mathbb{N}^\infty$ be a section of the factor map. Then $\mathcal{L}[E]$ is the C^*-algebra generated in $M(\mathcal{L}[\mathbf{1}])$ by $\big\{\mathcal{L}[\varphi(\gamma)] \,\big|\, \gamma \in \mathbb{N}^\infty/\!\sim\big\}$, and it is the closure of the span of the elementary tensors in this generating set.

Below we will prove that $\mathcal{L}[E]$ is a full host algebra for (S, σ), and so it is of some interest to explore its algebraic structure. From the reducing property of products, we already know that $\mathcal{L}[E]$ has the ideal $\mathcal{L}[\mathbf{1}]$ (we will show that it is proper), hence that it is not simple. However, it has in fact infinitely many proper ideals and each of the generating algebras $\mathcal{L}[\mathbf{n}]$ is contained in such an ideal:

Proposition III.5. *For the C^*-algebra $\mathcal{L}[E]$, we have the following:*

(i) *$\mathcal{L}[E]$ is nonseparable,*

(ii) *Define $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ to be the closed span of*

$$\big\{\mathcal{L}[\mathbf{q}]_0 \;\mid\; [\mathbf{q}] \leq [\mathbf{n}_\ell] \text{ for some } \ell = 1, \ldots, k\big\}.$$

Let $[\mathbf{p}] > [\mathbf{n}_\ell]$ strictly for all $\ell \in \{1, \ldots, k\}$, then $\mathcal{L}[\mathbf{p}] \cap \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k] = \{0\}$.

(iii) *$\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is a proper closed two sided ideal of $\mathcal{L}[E]$.*

(iv) *Define $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] := C^*\left(\mathcal{L}[\mathbf{n}_1] \cup \cdots \cup \mathcal{L}[\mathbf{n}_k]\right)$. Then $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \subset \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ and*

$$C^*\left(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]\right) \subseteq \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k], \quad \text{where} \quad (\mathbf{q}_j)_\ell = \min\big((\mathbf{n}_j)_\ell, (\mathbf{n}_{k+1})_\ell\big).$$

Proof. (i) $\mathcal{L}[E] \supset Q := \big\{E[\mathbf{n}]_1 := E^{(1)}_{n_1} \otimes E^{(2)}_{n_2} \otimes \cdots \;\mid\; \mathbf{n} \in \mathbb{N}^\infty\big\}$. If $\mathbf{n} \neq \mathbf{p}$, there is some k for which $E^{(k)}_{n_k} \neq E^{(k)}_{p_k}$ and as the approximate identity is linearly increasing, one of these must be larger than the other, so take $E^{(k)}_{n_k} > E^{(k)}_{p_k}$ strictly. Group the remaining parts of the tensor product together, i.e., write

$$E[\mathbf{n}]_1 = E^{(k)}_{n_k} \otimes A \quad \text{and} \quad E[\mathbf{p}]_1 = E^{(k)}_{p_k} \otimes B,$$

where A and B are projections, then choose a product representation $\pi = \pi_1 \otimes \pi_2$ in which π_1 is faithful on $\mathcal{L}_k$ and π_2 is faithful on the C^*-algebra generated by A and B. Thus there is a unit vector $\psi \in \mathcal{H}_1$ such that $\big\|\pi_1(E^{(k)}_{n_k})\psi\big\| = 1$ and $\pi_1(E^{(k)}_{p_k})\psi = 0$. For any unit vector $\varphi \in \mathcal{H}_2$ we get

$$\begin{aligned}\big\|E[\mathbf{n}]_1 - E[\mathbf{p}]_1\big\| &\geq \Big\|(\pi_1 \otimes \pi_2)\big(E^{(k)}_{n_k} \otimes A - E^{(k)}_{p_k} \otimes B\big)(\psi \otimes \varphi)\Big\| \\ &= \big\|\pi_1(E^{(k)}_{n_k})\psi \otimes \pi_2(A)\varphi\big\| = \big\|\pi_1(E^{(k)}_{n_k})\psi\big\| \cdot \big\|\pi_2(A)\varphi\big\| = \big\|\pi_2(A)\varphi\big\|\end{aligned}$$

and by letting φ range over the unit ball we get that $\big\|E[\mathbf{n}]_1 - E[\mathbf{p}]_1\big\| \geq \|A\| = 1$. Thus, since Q is uncountable and its elements far apart, $\mathcal{L}[E]$ cannot be separable.

(ii) Here we adapt the argument in (i) as follows. It suffices to show that for $\mathbf{q}_1, \ldots, \mathbf{q}_d$ with $\mathbf{q}_i \leq \mathbf{n}_j$ for some j, the norm distance between $\sum_{i=1}^{d} \mathcal{L}[\mathbf{q}_i]_0$ and any $C \in \mathcal{L}[\mathbf{p}]_0$ is always $\geq \|C\|$. Let $C \in \mathcal{L}[\mathbf{p}]_0$ be nonzero and consider a sum $\sum_{i=1}^{d} C_i$ with $C_i \in \mathcal{L}[\mathbf{q}_i]_0$ and $[\mathbf{p}] > [\mathbf{n}_j]$ for all j, which implies $[\mathbf{p}] > [\mathbf{q}_i]$ for all i. Choose an $M > 0$ large enough so that all C and C_i can be expressed in the form:

$$C_i = C_i^{(0)} \otimes E[\mathbf{n}_i]_M\,, \quad \text{for} \quad C_i^{(0)} \in \mathcal{L}^{(M-1)}\,.$$

Then by $[\mathbf{p}] > [\mathbf{q}_i]$ there is an entry of the tensor products, say for $j > M$, which consist only of elements of the approximate identity $(E_n^{(j)})_{n=1}^{\infty} \subset \mathcal{L}_j$ and for which $B > B_i$ for all i, where B (resp. B_i) is the j^{th} entry of C (resp. C_i). Denote the remaining parts of the tensor products by A (resp. A_i), i.e.,

$$C = A \otimes B, \quad C_i = A_i \otimes B_i, \quad \text{where} \quad B > B_i \;\forall\, i$$

and $B,\ B_i$ consist of commuting projections. Then $\|C - \sum_{i=1}^{d} C_i\| = \big\|A \otimes B - \sum_{i=1}^{d} (A_i \otimes B_i)\big\|$. Choose a product representation $\pi = \pi_1 \otimes \pi_2$ such that π_1 is faithful on $\mathcal{L}[\mathbf{p}]$ and π_2 is faithful on the C^*-algebra generated by $(E_n^{(j)})_{n=1}^{\infty} \subset \mathcal{L}_j$. Thus there is a unit vector $\varphi \in \mathcal{H}_{\pi_2}$ such that $\|\pi_2(B)\varphi\| = 1$ and $\pi_2(B_i)\varphi = 0$ for all i (which exists because $B > B_i$ for all i). Then we have for any unit vector $\psi \in \mathcal{H}_{\pi_1}$ that

$$\begin{aligned}\|C - \sum_{i=1}^{d} C_i\| &\geq \Big\|(\pi_1 \otimes \pi_2)\Big(A \otimes B - \sum_{i=1}^{d} A_i \otimes B_i\Big)(\psi \otimes \varphi)\Big\| \\ &= \|\pi_1(A)\psi \otimes \pi_2(B)\varphi\| = \|\pi_1(A)\psi\| \cdot \|\pi_2(B)\varphi\| = \|\pi_1(A)\psi\|\end{aligned}$$

and by letting ψ range over the unit ball of $\mathcal{H}_{\pi_1}$, we find that $\|C - \sum_{i=1}^{d} C_i\| \geq \|A\| = \|C\|$ since $\|B\| = 1$. This establishes the claim.

(iii) It is obvious from the reduction property $\mathcal{L}[\mathbf{n}] \cdot \mathcal{L}[\mathbf{p}] \subseteq \mathcal{L}[\mathbf{q}]$ for $q_j = \min(n_j, p_j)$, that $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is a two–sided ideal (hence a *–algebra). To see that it is proper, note that $[\mathbf{p}] > [\mathbf{n}_i]$ strictly for all i where $p_j = \max((\mathbf{n}_1)_j, \ldots, (\mathbf{n}_k)_j) + 1$. Thus, by (ii) we see that $\mathcal{L}[\mathbf{p}] \cap \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k] = \{0\}$ and hence that $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is proper.

(iv) $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \subset \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ because $\mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ is a C^*-algebra which contains all the generating elements $\mathcal{L}[\mathbf{n}_i]$ of $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$. Next we need to prove that

$$C^*\left(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]\right) \subseteq \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k], \quad \text{where} \quad (\mathbf{q}_j)_\ell = \min\big((\mathbf{n}_j)_\ell,\, (\mathbf{n}_{k+1})_\ell\big)\,.$$

By definition, $C^*\left(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]\right)$ is the closed linear span of monomials $\prod_{i=1}^{N} L_i$, where L_i can be either of the form $A_i B_i$ or $B_i A_i$, where $A_i \in \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ and $B_i \in \mathcal{L}[\mathbf{n}_{k+1}]$. So it suffices to show that

$$AB \in \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k] \quad \text{for} \quad A \in \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \quad \text{and} \quad B \in \mathcal{L}[\mathbf{n}_{k+1}]$$

(since then $BA \in \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k]$ by involution). Since $\mathcal{L}[\mathbf{n}]_0$ is dense in $\mathcal{L}[\mathbf{n}]$, it suffices to prove this for $A = A_1 A_2 \ldots A_p$ where $A_i = C_i \otimes E[\mathbf{n}_{k_i}]_{r_i+1}$ and $C_i \in \mathcal{L}^{(r_i)}$, $k_i \in \{1, \ldots, k\}$, and $B = D \otimes E[\mathbf{n}_{k+1}]_{r+1}$, where $D \in \mathcal{L}^{(r)}$. Now

$$A_p B = F \otimes E[\mathbf{q}_{k_p}]_{s+1} \in \mathcal{L}[\mathbf{q}_{k_p}]$$

for some $F \in \mathcal{L}^{(s)}$, $s \geq \max(r_p, r)$. Then

$$A_{p-1} A_p B = \big(C_{p-1} \otimes E[\mathbf{n}_{k_{p-1}}]_{r_{p-1}+1}\big)\big(F \otimes E[\mathbf{q}_{k_p}]_{s+1}\big) = G \otimes E[\mathbf{m}]_{t+1},$$

where $t \geq \max(r_{p-1}, s)$ and

$$\begin{aligned} m_i &= \min\big((\mathbf{n}_{k_{p-1}})_i,\, (\mathbf{q}_{k_p})_i\big) = \min\Big((\mathbf{n}_{k_{p-1}})_i,\, \min\big((\mathbf{n}_{k_p})_i,\, (\mathbf{n}_{k+1})_i\big)\Big) \\ &= \min\Big(\min\big((\mathbf{n}_{k_{p-1}})_i,\, (\mathbf{n}_{k+1})_i\big),\, \min\big((\mathbf{n}_{k_p})_i,\, (\mathbf{n}_{k+1})_i\big)\Big) = \min\big((\mathbf{q}_{k_{p-1}})_i,\, (\mathbf{q}_{k_p})_i\big) \end{aligned}$$

and so we have in fact that

$$A_{p-1} A_p B = \big(\widetilde{C} \otimes E[\mathbf{q}_{k_{p-1}}]_{t+1}\big)\big(\widetilde{F} \otimes E[\mathbf{q}_{k_p}]_{t+1}\big) \in \mathcal{L}[\mathbf{q}_{k_{p-1}}] \cdot \mathcal{L}[\mathbf{q}_{k_p}]$$

where $\widetilde{C}, \widetilde{F} \in \mathcal{L}^{(t)}$. Hence $A_{p-1} A_p B \in \mathcal{L}[\mathbf{q}_{k_{p-1}}, \mathbf{q}_{k_p}]$. We continue the process to get $AB = A_1 A_2 \ldots A_p B \in \mathcal{L}[\mathbf{q}_1, \ldots, \mathbf{q}_k]$. ■

For each strictly increasing sequence $\big([\mathbf{n}_1], [\mathbf{n}_2], \ldots\big) \subset \mathbb{N}^\infty/\sim$ we get from part (ii) a strictly increasing chain of proper ideals $\mathcal{J}_k := \mathcal{I}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$.

Now we want to prove our main theorem in this section.

Theorem III.6. *The monomorphism $\eta : S_\sigma \to U(M(\mathcal{L}[E]))$ from above, defined by*

$$\eta((s,t)) := t\delta_s \in \mathcal{A} \subset M(\mathcal{L}[E]),$$

is continuous with respect to the strict topology on $M(\mathcal{L}[E])$ and $\mathcal{L}[E]$ is a host algebra, i.e., the map

$$\eta^* : \mathrm{Rep}(\mathcal{L}[E], \mathcal{H}) \to \mathrm{Rep}((S, \sigma), \mathcal{H})$$

is injective. The range of η^* is exactly $\mathcal{R}(\mathcal{H})$.

Proof. First we show that η is continuous with respect to the strict topology on $M(\mathcal{L}[E])$. This implies that for each $\pi \in \mathrm{Rep}(\mathcal{L}[E], \mathcal{H})$ the representation $\widetilde{\pi} \in \mathrm{Rep}(\mathcal{A}, \mathcal{H})$ is regular, hence

$$\eta^*\big(\mathrm{Rep}(\mathcal{L}[E], \mathcal{H})\big) \subseteq \mathcal{R}(\mathcal{H}).$$

Since $\mathrm{im}(\eta)$ is bounded, it suffices to show that the set

$$\big\{L \in \mathcal{L}[E] \;\mid\; g \mapsto \eta(g)L \quad \text{is norm continuous in} \quad g \in S_\sigma\big\}$$

spans a dense subspace of $\mathcal{L}[E]$. This reduces the assertion to the corresponding result for the action of S_σ on $\mathcal{L}[\mathbf{n}]$ for each $\mathbf{n}$, which follows from the continuity of the corresponding map $S_\sigma \to M(\mathcal{L}[\mathbf{n}])$ (Proposition III.4).

To prove that η^* is injective we show that $\mathcal{A}$ separates $\mathrm{Rep}(\mathcal{L}[E],\mathcal{H})$ for all $\mathcal{H}$. Let $\pi \in \mathrm{Rep}(\mathcal{L}[E],\mathcal{H})$, then by Proposition III.4 we know that the values which $\widetilde{\pi}(\mathcal{A})$ takes on $\mathcal{H}_{\mathbf{n}}$ uniquely determine the values of $\pi\big(\mathcal{L}[\mathbf{n}]\big)$ on its essential subspace $\mathcal{H}_{\mathbf{n}}$, hence on all $\mathcal{H}$, as $\pi\big(\mathcal{L}[\mathbf{n}]\big)$ is zero on the orthogonal complement of $\mathcal{H}_{\mathbf{n}}$. This holds for all $\mathbf{n}$, hence $\widetilde{\pi}(\mathcal{A})$ uniquely determines the values of π on $\mathcal{L}[E]$, i.e., η^* is injective.

It remains to prove that $\eta^*\big(\mathrm{Rep}(\mathcal{L},\mathcal{H})\big) = \mathcal{R}(\mathcal{H})$. Start from a $\pi \in \mathrm{Rep}(\mathcal{A},\mathcal{H})$ which is regular. Then we have to show how to obtain a $\pi_0 \in \mathrm{Rep}\,\mathcal{L}[E]$ such that $\widetilde{\pi}_0\restriction\mathcal{A} = \pi$. Observe that π is regular on all $\mathcal{A}^{(n)}$, hence there are unique $\pi_n \in \mathrm{Rep}(\mathcal{L}^{(n)},\mathcal{H})$ which extend (on $\mathcal{H}$) to coincide with $\pi\restriction\mathcal{A}^{(n)}$ by the host algebra property of $\mathcal{L}^{(n)}$. For each $\mathbf{n}$ define the projections

$$\mathbb{E}^{\mathbf{n}}_k := \mathop{\mathrm{s-lim}}_{m\to\infty} \pi(E^{(k)}_{n_k})\cdots\pi(E^{(m)}_{n_m}) \quad \text{and} \quad \mathbb{E}^{\mathbf{n}} := \mathop{\mathrm{s-lim}}_{k\to\infty} \mathbb{E}^{\mathbf{n}}_k .$$

Now each $\pi_n(\mathcal{L}^{(n)})$ commutes with the projections $\mathbb{E}^{\mathbf{n}}_k$ for $k > n$, and in particular preserves the space $\mathcal{H}^{\mathbf{n}} := \mathbb{E}^{\mathbf{n}}\mathcal{H}$, and hence so does $\pi(\mathcal{A}^{(n)})$. Then by Proposition III.4 we know that we can define a (non-degenerate) representation $\pi^{\mathbf{n}}_0 : \mathcal{L}[\mathbf{n}] \to \mathcal{B}(\mathcal{H}^{\mathbf{n}})$ by

$$\pi^{\mathbf{n}}_0(L) = \pi_k(A_1 \otimes \cdots \otimes A_k)\,\mathbb{E}^{\mathbf{n}}_{k+1}$$

for $L = A_1 \otimes\cdots\otimes A_k \otimes E^{(k+1)}_{n_{k+1}} \otimes E^{(k+2)}_{n_{k+2}} \otimes \cdots \in \mathcal{L}[\mathbf{n}]$ such that $\widetilde{\pi}^{\mathbf{n}}_0\restriction\mathcal{A}$ is $\pi(\mathcal{A})$, restricted to $\mathcal{H}^{\mathbf{n}}$. We extend $\pi^{\mathbf{n}}_0$ to all of $\mathcal{H}$, by putting it to zero on the orthogonal complement of $\mathcal{H}^{\mathbf{n}}$. Note that

$$\mathbf{n} \leq \mathbf{m} \quad \Rightarrow \quad \mathcal{H}^{\mathbf{n}} \subseteq \mathcal{H}^{\mathbf{m}}.$$

We now argue that these representations $\pi^{\mathbf{n}}_0$ combine into a single representation of $\mathcal{L}[E]$. First, we want to extend by linearity the maps $\pi^{\mathbf{n}}_0 : \mathcal{L}[\mathbf{n}] \to \mathcal{B}(\mathcal{H})$ to define a linear map π_0 from the dense $*$-subalgebra $\mathcal{L}_0 \subset \mathcal{L}[E]$ to $\mathcal{B}(\mathcal{H})$, where we recall that $\mathcal{L}_0 := \sum_{\mathbf{n}\in\mathbb{N}^\infty} \mathcal{L}[\mathbf{n}]_0$ (finite sums).

This linear extension π_0 is possible if the sum of the spaces $\mathcal{L}[\mathbf{n}]_0$ is direct for different $\mathbf{n} \in \varphi\big(\mathbb{N}^\infty/\sim\big)$, i.e., if $0 = \sum_{k=1}^m B_k$ for $B_k \in \mathcal{L}[\mathbf{n}_k]_0$, where $\mathbf{n}_k \not\sim \mathbf{n}_\ell$ if $k \neq \ell$ implies that $B_k = 0$ for all k. Let us prove this implication, so assume $0 = \sum_{k=1}^m B_k$ as above. Choose an $M > 0$ large enough so that for all k, the B_k can be expressed in the form $B_k = A_k \otimes E[\mathbf{n}_k]_M$ for $A_k \in \mathcal{L}^{(M-1)}$, define the projections $P_k := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes E[\mathbf{1}]_k$ (there are $k-1$ factors of $\mathbb{1}$), and note that P_ℓ commutes with all B_k for $\ell \geq M$. In fact, for B_k as above, we have (simplifying notation to $\mathbf{n}_k = \mathbf{n}$):

$$B_k P_\ell = A_k \otimes E^{(M)}_{n_M} \otimes \cdots \otimes E^{(\ell-1)}_{n_{\ell-1}} \otimes E[\mathbf{1}]_\ell \in \mathcal{L}^{(\ell-1)} \otimes E[\mathbf{1}]_\ell$$

and so multiplication by P_ℓ for $\ell \geq M$ maps the B_k to elementary tensors of the form $A_k \otimes E^{(M)}_{n_M} \otimes \cdots \otimes E^{(\ell-1)}_{n_{\ell-1}}$ in $\mathcal{L}^{(\ell-1)}$ (after identifying $\mathcal{L}^{(\ell-1)} \otimes E[\mathbf{1}]_\ell$ with $\mathcal{L}^{(\ell-1)}$). Now a set of elementary tensors (in a finite tensor product) will be linearly independent if the entries in a fixed slot are linearly independent so it suffices to find $\ell > M$ such that the pieces $E^{(M)}_{n_M} \otimes \cdots \otimes E^{(\ell-1)}_{n_{\ell-1}}$ are linearly independent for $\mathbf{n} \in N := \big\{\mathbf{n}_k \mid k = 1, \ldots, m\big\}$. Since the approximate identities $(E^{(k)}_n)_{n=1}^\infty \subset \mathcal{L}_k$ consist of strictly increasing projections, their terms are linearly independent from which it follows that tensor products of these with distinct entries are linearly independent. Thus we only have to identify an ℓ large enough so that the portions of the sequences $\mathbf{n}_k$ between the entries M and ℓ can distinguish all the sequences in N, and this is always possible since the $\mathbf{n}_k$ are representatives of distinct equivalence classes in $\mathbb{N}^\infty/\!\sim$. Thus $\{B_1P_\ell, \ldots, B_mP_\ell\}$ is linearly independent for this ℓ, so $0 = \sum\limits_{k=1}^m B_kP_\ell$ implies that all $B_k = 0$. We conclude that the linear extension π_0 exists.

That π_0 respects involution is clear. To see that it is a homomorphism, consider the elementary tensors

$$L = A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes E[\mathbf{n}]_{k+1} \in \mathcal{L}[\mathbf{n}] \quad \text{and} \quad M = B_1 \otimes B_2 \otimes \cdots \otimes B_m \otimes E[\mathbf{p}]_{m+1} \in \mathcal{L}[\mathbf{p}]$$

where $m > k$ and $\mathbf{n} \not\sim \mathbf{p} \in \mathbb{N}^\infty$. Then

$$\begin{aligned}
\pi_0(L)\,\pi_0(M) &= \pi_k(A_1 \otimes \cdots \otimes A_k)\,\mathbb{E}^{\mathbf{n}}_{k+1}\pi_m(B_1 \otimes \cdots \otimes B_m)\,\mathbb{E}^{\mathbf{p}}_{m+1} \\
&= \pi_m\big(A_1 \otimes \cdots \otimes A_k \otimes E^{(k+1)}_{n_{k+1}} \otimes \cdots \otimes E^{(m)}_{n_m}\big)\,\mathbb{E}^{\mathbf{n}}_{m+1}\pi_m(B_1 \otimes \cdots \otimes B_m)\,\mathbb{E}^{\mathbf{p}}_{m+1} \\
&= \pi_m\big(A_1B_1 \otimes \cdots \otimes A_kB_k \otimes E^{(k+1)}_{n_{k+1}}B_{k+1} \cdots \otimes E^{(m)}_{n_m}B_m\big)\,\mathbb{E}^{\mathbf{n}}_{m+1}\,\mathbb{E}^{\mathbf{p}}_{m+1}\,.
\end{aligned}$$

Now recall that the operator product is jointly continuous on bounded sets in the strong operator topology, hence

$$\begin{aligned}
\mathbb{E}^{\mathbf{n}}_k\,\mathbb{E}^{\mathbf{p}}_k &= \operatorname*{s-lim}_{m\to\infty} \pi(E^{(k)}_{n_k}) \cdots \pi(E^{(m)}_{n_m}) \cdot \operatorname*{s-lim}_{r\to\infty} \pi(E^{(k)}_{p_k}) \cdots \pi(E^{(r)}_{p_r}) \\
&= \operatorname*{s-lim}_{m\to\infty} \pi(E^{(k)}_{n_k}) \cdots \pi(E^{(m)}_{n_m})\pi(E^{(k)}_{p_k}) \cdots \pi(E^{(m)}_{p_m}) \\
&= \operatorname*{s-lim}_{m\to\infty} \pi(E^{(k)}_{q_k}) \cdots \pi(E^{(m)}_{q_m}) = \mathbb{E}^{\mathbf{q}}_k
\end{aligned}$$

where $q_j := \min(n_j,\, p_j)$. Thus we get exactly that $\pi_0(L)\,\pi_0(M) = \pi_0(LM)$.

We now verify that π_0 is bounded. For this, we first need to prove the following:

Claim: *Recall that* $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] = C^*\left(\mathcal{L}[\mathbf{n}_1] \cup \cdots \cup \mathcal{L}[\mathbf{n}_k]\right)$. *Then for each* $k \geq 1$ *and* k*-tuple* $(\mathbf{n}_1, \ldots, \mathbf{n}_k)$ *such that* $\mathbf{n}_k \not\sim \mathbf{n}_\ell$ *if* $k \neq \ell$ *the map* π_0 *on* $\mathcal{L}_0 \cap \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ *extends to a representation of the* C^**-algebra* $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$.

Proof: Note that the claim implies the compatibility of the representations, i.e., on intersections $\mathcal{L}[\mathbf{p}_1, \ldots, \mathbf{p}_\ell] \cap \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$, the representations produced by π_0 on $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$ and

$\mathcal{L}[\mathbf{p}_1,\ldots,\mathbf{p}_\ell]$ coincide. This is because π_0 is given as a consistent map on the dense space $\mathcal{L}_0$. We now prove the claim by induction on k. We already have by definition that π_0 is the representation $\pi^{\mathbf{n}}$ on $\mathcal{L}[\mathbf{n}]$ for each $\mathbf{n}$, hence the claim is true for $k=1$. Assume the claim is true for all values of k up to a fixed $k\geq 1$, then we now prove it for $k+1$. Observe that $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$ contains the closed two–sided ideals

$$\mathcal{J}_1 := C^*\left(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\cdot\mathcal{L}[\mathbf{n}_{k+1}]\right)\subset\mathcal{J}_2\cap\mathcal{J}_3,$$

where
$$\mathcal{J}_2 := \mathcal{J}_1+\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\quad\text{and}\quad\mathcal{J}_3 := \mathcal{J}_1+\mathcal{L}[\mathbf{n}_{k+1}]$$

and that $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]=\mathcal{J}_2+\mathcal{J}_3$. We will prove below that $\mathcal{J}_1$ is proper (hence that the ideal structure above is nontrivial). Consider the factorization $\xi:\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\to\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]/\mathcal{J}_1$. Then

$$\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]\big)=\xi\big(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)+\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$$

and $\xi(\mathcal{J}_2)\cdot\xi(\mathcal{J}_3)=0$. If $\mathcal{J}_1$ is *not* proper, then

$$\mathcal{L}[\mathbf{n}_{k+1}]\subset\mathcal{J}_1\supset\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k].$$

By Proposition III.5(iv), we have that

$$\mathcal{J}_1\subset\mathcal{L}[\mathbf{q}_1,\ldots,\mathbf{q}_k]\subset\mathcal{I}[\mathbf{q}_1,\ldots,\mathbf{q}_k]\quad\text{for}\quad(\mathbf{q}_j)_\ell=\min\big((\mathbf{n}_j)_\ell,\,(\mathbf{n}_{k+1})_\ell\big),$$

and hence $\mathcal{L}[\mathbf{n}_{k+1}]\subset\mathcal{J}_1\subset\mathcal{I}[\mathbf{q}_1,\ldots,\mathbf{q}_k]$. Thus, by Proposition III.5(ii) we conclude that $[\mathbf{n}_{k+1}]$ cannot be strictly greater than all the $[\mathbf{q}_i]$, i.e., there is one member of the set $\{\mathbf{q}_1,\ldots,\mathbf{q}_k\}$, say $\mathbf{q}_j$, which satisfies $[\mathbf{q}_j]=[\mathbf{n}_{k+1}]$, and so by definition of $\mathbf{q}_j$, we have that eventually $(\mathbf{n}_{k+1})_\ell=\min\big((\mathbf{n}_j)_\ell,\,(\mathbf{n}_{k+1})_\ell\big)$, i.e., $[\mathbf{n}_j]\geq[\mathbf{n}_{k+1}]$.

Likewise, the inclusion $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\subset C^*\left(\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\cdot\mathcal{L}[\mathbf{n}_{k+1}]\right)=\mathcal{J}_1$ implies that no $\mathbf{n}_j$, $j=1,\ldots,k$, is reduced through multiplication by $\mathbf{n}_{k+1}$, i.e., eventually $(\mathbf{n}_j)_\ell=\min\big((\mathbf{n}_j)_\ell,\,(\mathbf{n}_{k+1})_\ell\big)$ for all j, i.e., $[\mathbf{n}_j]\leq[\mathbf{n}_{k+1}]$. So, together with the previous inequality, we see that there must be a $j\in\{1,\ldots,k\}$ such that $[\mathbf{n}_j]=[\mathbf{n}_{k+1}]$. This contradicts the initial assumption that all $[\mathbf{n}_\ell]$ are distinct, and so $\mathcal{J}_1$ must be proper.

Now consider π_0 on $\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_{k+1}]$. By the induction assumption, π_0 on

$$\mathcal{L}_0\cap\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$$

is the restriction of a representation on $\mathcal{L}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$,- we denote the projection onto its essential subspace by $\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$. Note that $\mathbb{E}[\mathbf{n}_{k+1}]$ commutes with $\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]$ because it commutes with all the generating elements $\pi_0(L_i)=\pi^{\mathbf{n}_i}(L_i)$, $L_i\in\mathcal{L}[\mathbf{n}_i]$. Thus we have an orthogonal decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4$, where

$$\mathcal{H}_1 := \mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\mathbb{E}[\mathbf{n}_{k+1}]\mathcal{H},\qquad\mathcal{H}_2 := \mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big(\mathbb{1}-\mathbb{E}[\mathbf{n}_{k+1}]\big)\mathcal{H}$$
$$\mathcal{H}_3 := \mathbb{E}[\mathbf{n}_{k+1}]\big(\mathbb{1}-\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)\mathcal{H},\qquad\mathcal{H}_4 := \big(\mathbb{1}-\mathbb{E}[\mathbf{n}_{k+1}]\big)\big(\mathbb{1}-\mathbb{E}[\mathbf{n}_1,\ldots,\mathbf{n}_k]\big)\mathcal{H}$$

and π_0 preserves these subspaces. Now by Proposition III.5(iv) and the induction assumption, π_0 extends from the $\mathcal{L}_0 \cap \mathcal{J}_1$ to a representation on $\mathcal{J}_1$, and as $\mathcal{J}_1 = C^*\left(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k] \cdot \mathcal{L}[\mathbf{n}_{k+1}]\right)$, the essential projection for $\pi_0 \restriction \mathcal{J}_1$ is $\mathbb{E}[\mathbf{n}_1, \ldots, \mathbf{n}_k]\mathbb{E}[\mathbf{n}_{k+1}]$, i.e., its essential subspace is $\mathcal{H}_1$. But since $\mathcal{J}_1$ is a closed two–sided ideal of $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$, its non-degenerate representations extend uniquely to $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$. Thus on $\mathcal{H}_1$, π_0 extends from $\mathcal{L}_0 \cap \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$ to a representation on $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$.

Next observe that on $\mathcal{H}_1^\perp = \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ we have $\{0\} = \pi_0(\mathcal{J}_1)$. We show that one can define a consistent representation of $\xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]\big)$ by $\rho(\xi(A)) := \pi_0(A) \restriction \mathcal{H}_1^\perp$, for $A \in \mathcal{L}[\mathbf{n}_{k+1}] + \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$, using the structure of $\xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]\big)$ above. First observe that ρ is well-defined on $\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$ and $\xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]\big)$ separately, because if $A_1 - A_2 \in \mathcal{J}_1$, then $\pi_0(A_1 - A_2) \restriction \mathcal{H}_1^\perp = 0$. Next, ρ is well-defined on the set $\xi\big(\mathcal{L}[\mathbf{n}_{k+1}] + \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]\big)$ by the induction assumption, and the consistency of the extensions of π_0. To see that ρ is well-defined on the algebra $\xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]\big) = \xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]\big) + \xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$, it suffices by the direct sum decomposition to check it on $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ separately. On $\mathcal{H}_2$, π_0 vanishes on $\mathcal{L}[\mathbf{n}_{k+1}]$, so since $\xi\big(\mathcal{L}[\mathbf{n}_{k+1}]\big)$ is an ideal of $\xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]\big)$ (and $\xi(\mathcal{J}_2) \cdot \xi(\mathcal{J}_3) = \{0\}$), it follows that we can extend $\rho(\xi(A)) \restriction \mathcal{H}_2$ by linearity, i.e., $\rho(\xi(A) + \xi(B)) = \rho(\xi(A))$ for $A \in \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$, $B \in \mathcal{L}[\mathbf{n}_{k+1}]$ to define a representation on $\xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]\big)$. Likewise, on $\mathcal{H}_3$, π_0 vanishes on $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_k]$, so we can show ρ defines a representation of $\xi\big(\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]\big)$ and on $\mathcal{H}_4$, ρ is zero. Then ρ lifts to a representation of $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$ on $\mathcal{H}_1^\perp$ which coincides with π_0 on $\mathcal{L}_0 \cap \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$. Taking the direct sum of this with the representation we obtained on $\mathcal{H}_1$, produces a representation of $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$ on all $\mathcal{H}$ which coincides with π_0 on $\mathcal{L}_0 \cap \mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_{k+1}]$. Thus, we have proven the claim for $k+1$, which completes the induction. ▼

That π_0 is bounded on $\mathcal{L}_0$ now follows immediately from the claim, because any $A \in \mathcal{L}_0$ is of the form $A = \sum_{k=1}^{m} B_k$ for $B_k \in \mathcal{L}[\mathbf{n}_k]_0$, where $\mathbf{n}_k \not\sim \mathbf{n}_\ell$ if $k \neq \ell$. But this is an element of $\mathcal{L}[\mathbf{n}_1, \ldots, \mathbf{n}_m]$ and by the claim π_0 extends as a representation to it, hence $\|\pi_0(A)\| \leq \|A\|$. We conclude that π_0 is a bounded representation, hence extends to all of $\mathcal{L}[E]$. To see that π_0 is non-degenerate, recall that $\{E_n^{(k)}\} \subset \mathcal{L}_k$ is an approximate identity of increasing projections. Thus we can find a sequence $\mathbf{n}$ such that $\operatorname*{s-lim}_{m\to\infty} \pi(E_{n_m}^{(m)}) = \mathbb{1}$, and hence $\mathbb{E}^{\mathbf{n}} = \mathbb{1}$ by $\pi(E_{n_m}^{(m)}) \leq \mathbb{E}^{\mathbf{n}} \leq \mathbb{1}$ for all m. Since the essential subspace of $\pi_0 \restriction \mathcal{L}[\mathbf{n}]$ is $\mathbb{E}^{\mathbf{n}}\mathcal{H}$, it follows that π_0 is non-degenerate. It then follows from Proposition III.4 applied to $\mathcal{L}[\mathbf{n}]$ that $\widetilde{\pi}_0 \restriction \mathcal{A} = \pi$. ■

Finally, we apply the structures above to produce a direct integral of regular representations into irreducible regular representations. First observe that given any representation $\pi \in \mathrm{Rep}((S,\sigma),\mathcal{H})$, where $\mathcal{H}$ is separable, then as $(E_n)_{n\in\mathbb{N}}$ is an approximate identity for $\mathcal{K}(\ell^2(\mathbb{N}))$, there is a sequence $\mathbf{n}$ such that $\operatorname*{s-lim}_{k\to\infty} \operatorname*{s-lim}_{\ell\to\infty} \pi(E_{n_k}^{(k)}) \cdots \pi(E_{n_\ell}^{(\ell)}) = \mathbb{1}$, and thus by Proposition III.4 there is a unique $\pi_0 \in \mathrm{Rep}(\mathcal{L}[\mathbf{n}],\mathcal{H})$ such that $\eta^*\pi_0 = \pi$. Fix a choice of maxi-

mally commutative subalgebra $\mathcal{C} \subset \pi_0(\mathcal{L}[\mathbf{n}])'$. Then, since $\mathcal{L}[\mathbf{n}]$ is separable, there is an extremal decomposition of π_0 (cf. [BR03] Corollary 4.4.8), i.e., there is a standard measure space (Z,μ) with μ a positive bounded measure, a measurable family $z \to \mathcal{H}(z)$ of Hilbert spaces, a measurable family $z \to \pi_z \in \mathrm{Rep}(\mathcal{L}[\mathbf{n}],\mathcal{H}(z))$ of representations which are almost all irreducible and a unitary $U : \mathcal{H} \to \int_Z^\oplus \mathcal{H}(z)\,d\mu(z)$ such that $U\mathcal{C}U^{-1}$ is the diagonizable operators, and

$$U\pi_0(A)U^{-1} = \int_Z^\oplus \pi_z(A)\,d\mu(z) \;\; \forall\, A \in \mathcal{L}[\mathbf{n}].$$

Then for $\psi,\, \varphi \in \int_Z^\oplus \mathcal{H}(z)\,d\mu(z)$ with decompositions $\psi = \int_Z^\oplus \psi_z\,d\mu(z)$ and $\varphi = \int_Z^\oplus \varphi_z\,d\mu(z)$, we have for $s \in S$ and any countable approximate identity (F_k) of $\mathcal{L}[\mathbf{n}]$ that

$$\begin{aligned}
&\big(\varphi,\, U\pi(s)U^{-1}\psi\big) = \big(\varphi,\, U\eta^*\pi_0(s)U^{-1}\psi\big) = \lim_{k\to\infty} \big(\varphi,\, U\pi_0(\delta_s F_k)U^{-1}\psi\big)\\
&= \lim_{k\to\infty} \int_Z \big(\varphi_z,\, \pi_z(\delta_s F_k)\,\psi_z\big)\,d\mu(z) = \int_Z \lim_{k\to\infty} \big(\varphi_z,\, \pi_z(\delta_s F_k)\,\psi_z\big)\,d\mu(z)\\
&= \int_Z \big(\varphi_z,\, \eta^*\pi_z(s)\,\psi_z\big)\,d\mu(z) = \Big(\varphi,\, \int_Z^\oplus \eta^*\pi_z(s)\,d\mu(z)\psi\Big),
\end{aligned}$$

where the usage of the Dominated Convergence Theorem in the second line is justified by $\big|\big(\varphi_z,\, \pi_z(\delta_s F_k)\,\psi_z\big)\big| \le \|\varphi_z\|\|\psi_z\|$ as both of $z \to \varphi_z$ and $z \to \psi_z$ are square integrable w.r.t. μ. Hence

$$U\pi(s)U^{-1} = \int_Z^\oplus \eta^*\pi_z(s)\,d\mu(z) \;\; \forall\, s \in S.$$

Since η^* preserves irreducibility, almost all $\eta^*\pi_z$ are irreducible, and hence we obtain the promised decomposition.

Appendix. Host algebras and the strict topology

Lemma A.1. *Let X be a locally compact space.*

(a) *On each bounded subset of $M(C_0(X)) \cong C_b(X)$, the strict topology coincides with the topology of compact convergence, i.e., the compact open topology. This holds in particular for the subgroup $C(X,\mathbb{T}) \cong U(C_b(X))$.*

(b) *A unital $*$-subalgebra $S \subseteq C_b(X)$ is strictly dense if and only if it separates the points of X.*

Proof. (a) ([Bl98, Ex. 12.1.1(b)]) Let $\mathcal{B} \subseteq C_b(X)$ be a bounded subset with $\|f\| \le C$ for each $f \in \mathcal{B}$. For each $\varphi \in C_0(X)$ and $\varepsilon > 0$ we now find a compact subset $K \subseteq X$ with $|\varphi| \le \varepsilon$ outside K. For $f_i \to f$ in $\mathcal{B}$ with respect to the compact open topology, we then have

$$\|(f - f_i)\varphi\| \le \|(f - f_i)|_K\|\|\varphi\| + \varepsilon\|f - f_i\| \le \varepsilon\|\varphi\| + 2\varepsilon C$$

for sufficiently large i. Therefore the maps $\mathcal{B} \to C_0(X), f \mapsto f\varphi$ are continuous if $\mathcal{B}$ carries the compact open topology. This means that the strict topology on $\mathcal{B}$ is coarser than the compact open topology.

If, conversely, $K \subseteq X$ is a compact subset and $h \in C_0(X)$ with $h|_K = 1$, then

$$\|(f - f_i)|_K\| \leq \|(f - f_i)h\|$$

shows that the strict topology on $C_b(X)$ is finer than the compact open topology. This proves (a).

(b) If S is strictly dense, then it obviously separates the points of X because the point evaluations are strictly continuous.

Suppose, conversely, that S separates the points of X. Replacing S by its norm closure, we may w.l.o.g. assume that S is norm closed. Let $K \subseteq X$ be compact. Since S separates the points of K, the Stone-Weierstraß Theorem implies that $S|_K = C(K)$. For any $f \in C_b(X)$ we therefore find some $f_K \in S$ with $\|f_K\| \leq 2\|f\|$ and $f_K|_K = f|_K$ because the restriction map is a quotient morphism of C^*-algebras. Since the net (f_K) is bounded and converges to f in the compact open topology, (a) implies that it also converges in the strict topology. Therefore S is strictly dense in $C_b(X)$. ■

Tensor products of C^*-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\mathcal{A} \otimes \mathcal{B}$ their spatial C^*-tensor product (defined by the minimal cross norm) ([Fi96]), which is a suitable completion of the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$, turning it into a C^*-algebra. We then have homomorphisms

$$i_{\mathcal{A}}: M(\mathcal{A}) \to M(\mathcal{A} \otimes \mathcal{B}), \quad i_{\mathcal{B}}: M(\mathcal{B}) \to M(\mathcal{A} \otimes \mathcal{B}),$$

uniquely determined by

$$i_{\mathcal{A}}(\varphi)(A \otimes B) = (\varphi \cdot A) \otimes B, \quad i_{\mathcal{B}}(\varphi)(A \otimes B) = A \otimes (\varphi \cdot B).$$

Moreover, for each complex Hilbert space $\mathcal{H}$, we have

$$\mathrm{Rep}(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}) \cong \{(\alpha, \beta) \in \mathrm{Rep}(\mathcal{A}, \mathcal{H}) \times \mathrm{Rep}(\mathcal{B}, \mathcal{H}) : [\alpha(\mathcal{A}), \beta(\mathcal{B})] = \{0\}\}.$$

This correspondence is established by assigning to each pair (α, β) with commuting range the representation

$$\pi := \alpha \otimes \beta : \mathcal{A} \otimes \mathcal{B} \to B(\mathcal{H}), \quad a \otimes b \mapsto \alpha(a)\beta(b).$$

Note that this representation of $\mathcal{A} \otimes \mathcal{B}$ is non-degenerate if α and β are non-degenerate.

Lemma A.2. *The following assertions hold for the embedding* $i_{\mathcal{A}}\colon M(\mathcal{A}) \to M(\mathcal{A}\otimes\mathcal{B})$*:*

(1) *The map*

$$i_{\mathcal{A}}^{-1}\colon M(\mathcal{A})\otimes \mathbf{1} \to M(\mathcal{A}), \quad m\otimes \mathrm{id}_{\mathcal{B}} \mapsto m$$

is continuous with respect to the strict topology on its domain obtained from $\mathcal{A}\otimes\mathcal{B}$ *and the strict topology on its range obtained from* $\mathcal{A}$*.*

(2) *Its restriction to bounded subsets is a homeomorphism.*

(3) $i_{\mathcal{A}}(\mathcal{A})$ *is dense in* $M(\mathcal{A})\otimes\mathbf{1}$ *with respect to the strict topology on* $M(\mathcal{A}\otimes\mathcal{B})$*.*

Proof. (1) The strict topology on $M(\mathcal{A})$ is defined by the seminorms

$$p_a(m) = \|m\cdot a\| + \|a\cdot m\|,$$

satisfying $p_a \circ i_{\mathcal{A}}^{-1} = p_{a\otimes\mathbf{1}}$, which shows immediately that $i_{\mathcal{A}}^{-1}$ is continuous.

(2) Since the embedding $i_{\mathcal{A}}$ is isometric, it suffices to show that for each bounded subset $\mathcal{M}\subseteq M(\mathcal{A})$, the restriction of $i_{\mathcal{A}}$ to $\mathcal{M}$ is continuous. Since $i_{\mathcal{A}}$ is linear, it suffices to show that for each bounded net (M_ν) with $\lim M_\nu = 0$ in the strict topology of $M(\mathcal{A})$, we also have $\lim i_{\mathcal{A}}(M_\nu) = 0$ in $M(\mathcal{A}\otimes\mathcal{B})$. For $A\in\mathcal{A}$ and $B\in\mathcal{B}$ we have

$$\|M_\nu(A\otimes B)\| = \|M_\nu A\otimes B\| = \|M_\nu A\|\|B\| \to 0$$

and likewise $(A\otimes B)M_\nu \to 0$. Since the elementary tensors span a dense subset of $A\otimes B$, the boundedness of the net (M_ν) implies that $i_{\mathcal{A}}(M_\nu)\to 0$ holds in the strict topology of $M(\mathcal{A}\otimes\mathcal{B})$ (cf. Wegge–Olsen [WO93], Lemma 2.3.6).

(3) Let $\{E_\alpha\}$ be any approximate identity of $\mathcal{A}$, satisfying $\|E_\alpha\|\leq 1$. Then for any $A\in M(\mathcal{A})$, the net $\{AE_\alpha\}\subset M(\mathcal{A})$ is bounded by $\|A\|$ and converges to A in the strict topology of $M(\mathcal{A})$, and hence in the strict topology of $M(\mathcal{A}\otimes\mathcal{B})$ by (2). This proves (3). ∎

Lemma A.3. *For each non-degenerate representation* $\pi\in\mathrm{Rep}(\mathcal{A}\otimes\mathcal{B},\mathcal{H})$ *the representations* $\pi_1(a) := \widetilde{\pi}(a\otimes\mathbf{1})$ *and* $\pi_2(b) := \widetilde{\pi}(\mathbf{1}\otimes b)$ *are non-degenerate, where* $\widetilde{\pi}$ *denotes the unique extension of* π *from* $\mathcal{A}\otimes\mathcal{B}$ *to* $M(\mathcal{A}\otimes\mathcal{B})$*. Moreover, the corresponding extensions* $\widetilde{\pi}_1\in\mathrm{Rep}(M(\mathcal{A}),\mathcal{H})$ *and* $\widetilde{\pi}_2\in\mathrm{Rep}(M(\mathcal{B}),\mathcal{H})$ *from* π_1*,* π_2 *on* $\mathcal{A}$*,* $\mathcal{B}$ *resp., satisfy*

$$\widetilde{\pi}_1 = \widetilde{\pi}\circ i_{\mathcal{A}} \quad \textit{and} \quad \widetilde{\pi}_2 = \widetilde{\pi}\circ i_{\mathcal{B}}.$$

In particular, the representations $\widetilde{\pi}\circ i_{\mathcal{A}}$ *and* $\widetilde{\pi}\circ i_{\mathcal{B}}$ *are continuous with respect to the strict topology on* $M(\mathcal{A})$*,* $M(\mathcal{B})$ *resp., and the the topology of pointwise convergence on* $B(\mathcal{H})$*.*

Proof. To see that π_1 is non-degenerate, we observe that for $a\otimes b\in\mathcal{A}\otimes\mathcal{B}$ we have $\pi(a\otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$, so that any vector annihilated by $\pi_1(\mathcal{A})$ is also annihilated by $\mathcal{A}\otimes\mathcal{B}$, hence zero. The same argument proves non-degeneracy of π_2.

For $m\in M(\mathcal{A})$, we have

$$\widetilde{\pi}(m\otimes\mathbf{1})\pi_1(a) = \widetilde{\pi}(m\otimes\mathbf{1})\widetilde{\pi}(a\otimes\mathbf{1}) = \widetilde{\pi}(ma\otimes\mathbf{1}) = \pi_1(ma) = \widetilde{\pi}_1(m)\widetilde{\pi}_1(a),$$

so that the non-degeneracy of π_1 implies $\widetilde{\pi} \circ i_{\mathcal{A}} = \widetilde{\pi}_1$, and likewise $\widetilde{\pi} \circ i_{\mathcal{B}} = \widetilde{\pi}_2$.

The last assertion follows from the general fact that for a non-degenerate representation of $\mathcal{A}$, the corresponding extension to $M(\mathcal{A})$ is continuous with respect to the strict topology on $M(\mathcal{A})$ and the topology of pointwise convergence on $B(\mathcal{H})$; similary for $\mathcal{B}$. ∎

Lemma A.4. *Let G_1, G_2 be topological groups and suppose that $(\mathcal{A}_1, \eta_1)$, resp., $(\mathcal{A}_2, \eta_2)$ are full host algebras for G_1, resp., G_2. Then*

$$\eta \colon G_1 \times G_2 \to M(\mathcal{A}_1 \otimes \mathcal{A}_2), \quad (g_1, g_2) \mapsto i_{\mathcal{A}_1}(\eta_1(g_1)) i_{\mathcal{A}_2}(\eta_2(g_2))$$

defines a full host algebra of $G_1 \times G_2$.

Proof. This follows from the observation that unitary representations of the direct product group $G := G_1 \times G_2$ can be viewed as pairs of commuting representations $\pi_j \colon G_j \to U(\mathcal{H})$, and we have the same picture on the level of non-degenerate representations of C^*-algebras. We only have to observe that both pictures are compatible. In fact, let π_j be commuting unitary representations of G_j, $j = 1, 2$, and $\widetilde{\pi}_j$ the corresponding representations of the host algebras $\mathcal{A}_j$. Then we have

$$\big(\eta^*(\widetilde{\pi}_1 \otimes \widetilde{\pi}_2)\big)(g_1, g_2) = (\widetilde{\pi}_1 \otimes \widetilde{\pi}_2)(\eta_1(g_1) \otimes \eta_2(g_2)) = \widetilde{\pi}_1(\eta_1(g_1)) \widetilde{\pi}_2(\eta_2(g_2)) = \pi_1(g_1)\pi_2(g_2). \quad ∎$$

Corollary A.7 below provides a converse to this lemma.

Ideals of multiplier algebras

Let $\mathcal{A}$ be a C^*-algebra and $M(\mathcal{A})$ its multiplier algebra. We are interested in the relation between the ideals of $\mathcal{A}$ and $M(\mathcal{A})$.

Lemma A.5. (a) *Each strictly closed ideal $J \subseteq M(\mathcal{A})$ coincides with the strict closure of the ideal $J \cap \mathcal{A}$ of $\mathcal{A}$, which is norm-closed.*

(b) *For each norm closed ideal $I \trianglelefteq \mathcal{A}$, its strict closure $\widetilde{I}$ satisfies $\widetilde{I} \cap \mathcal{A} = I$.*

(c) *The map $J \mapsto J \cap \mathcal{A}$ induces a bijection from the set of strictly closed ideals of $M(\mathcal{A})$ onto the set of norm-closed ideals of $\mathcal{A}$.*

Proof. (a) Let $(u_i)_{i \in I}$ be an approximate identity in $\mathcal{A}$ and $\mu \in J$. Then $\mu u_i \in J \cap \mathcal{A}$ converges to μ in the strict topology, and the assertion follows. Since on $\mathcal{A}$ the norm topology is finer than the strict topology, the ideal $J \cap \mathcal{A}$ of $\mathcal{A}$ is norm-closed.

(b) The ideal I is automatically $*$-invariant ([Dix64], Prop. 1.8.2), so that $\mathcal{A}/I$ is a C^*-algebra. Let $q \colon \mathcal{A} \to \mathcal{A}/I$ denote the quotient homomorphism. The existence of an approximate identity in $\mathcal{A}$ implies that I is invariant under the left and right action of the multiplier algebra, so that we obtain a natural homomorphism

$$M(q) \colon M(\mathcal{A}) \to M(\mathcal{A}/I),$$

which is strictly continuous ([Bu68, Prop. 3.8]). Then $\widetilde{I} := \ker M(q) \trianglelefteq M(\mathcal{A})$ is a strictly closed ideal satisfying $\widetilde{I} \cap \mathcal{A} = I$, and (a) implies that $\widetilde{I}$ is the strict closure of I.

(c) follows from (a) and (b). ■

The following proposition shows that for each closed normal subgroup N of a topological group G with a host algebra, the quotient group G/N also has a host algebra:

Proposition A.6. *Let G be a topological group and suppose that $\mathcal{A}$ is a host algebra for G with respect to the homomorphism*

$$\eta_G : G \to M(\mathcal{A}).$$

Let $N \trianglelefteq G$ be a closed normal subgroup, $\widetilde{I}_N \trianglelefteq M(\mathcal{A})$ the strictly closed ideal generated by $\eta_G(N) - \mathbf{1}$, and $I_N := \mathcal{A} \cap \widetilde{I}_N$. Then η_G factors through a homomorphism

$$\eta_{G/N} : G/N \to M(\mathcal{A}/I_N),$$

turning $\mathcal{A}/I_N$ into a host algebra for the quotient group G/N. If, in addition, $\mathcal{A}$ is a full host algebra of G, then $\mathcal{A}/I_N$ is a full host algebra of G/N.

Proof. If π is a unitary representation of G, then we write $\pi_{\mathcal{A}}$ for the corresponding representation of $\mathcal{A}$ and $\widetilde{\pi}_{\mathcal{A}}$ for the extension to $M(\mathcal{A})$ with $\widetilde{\pi}_{\mathcal{A}} \circ \eta_G = \pi$. Further, let $q_G : G \to G/N$ denote the quotient map.

We consider the C^*-algebra $\mathcal{B} := \mathcal{A}/I_N$ and recall that the quotient morphism $q : \mathcal{A} \to \mathcal{B}$ induces a strictly continuous morphism $M(q) : M(\mathcal{A}) \to M(\mathcal{B})$ ([Bu68, Prop. 3.8]). In view of $I_N = \ker q = (\ker M(q)) \cap \mathcal{A}$, Lemma A.5 implies that $\ker M(q) = \widetilde{I}_N$.

Next we observe that $\eta_G(N) - \mathrm{id}_{\mathcal{A}} \subseteq \widetilde{I}_N$ implies that N acts by trivial multipliers on the algebra $\mathcal{B} = \mathcal{A}/I_N$. We therefore obtain a group homomorphism

$$\eta_{G/N} : G/N \to U(M(\mathcal{B})) \quad \text{with} \quad \eta_{G/N} \circ q_G = M(q) \circ \eta_G.$$

To see that $\eta_{G/N}$ turns $\mathcal{B}$ into a host algebra for the quotient group G/N, we first note that every non-degenerate representation $\pi : \mathcal{B} \to B(H)$ can be viewed as a non-degenerate representation $\pi_{\mathcal{A}} : \mathcal{A} \to B(H)$ with $\pi_{\mathcal{A}} := \pi \circ q$. The corresponding representations of the multiplier algebras satisfy

$$\widetilde{\pi} \circ M(q) = \widetilde{\pi}_{\mathcal{A}} : M(\mathcal{A}) \to B(H).$$

This leads to

$$\widetilde{\pi} \circ \eta_{G/N} \circ q_G = \widetilde{\pi} \circ M(q) \circ \eta_G = \widetilde{\pi}_{\mathcal{A}} \circ \eta_G,$$

showing that the unitary representation $\widetilde{\pi} \circ \eta_{G/N}$ of G/N is continuous. We thus obtain a map

$$\eta^*_{G/N} : \mathrm{Rep}(\mathcal{B}) \to \mathrm{Rep}(G/N), \quad \pi \mapsto \widetilde{\pi} \circ \eta_{G/N}.$$

If two representations π and γ of $\mathcal{B}$ lead to the same representation of G/N, i.e.,

$$\eta_{G/N}^*(\pi) = \widetilde{\pi} \circ \eta_{G/N} = \widetilde{\gamma} \circ \eta_{G/N} = \eta_{G/N}^*(\gamma),$$

then the corresponding representations of G coincide, i.e., $\widetilde{\pi}_{\mathcal{A}} \circ \eta_G = \widetilde{\gamma}_{\mathcal{A}} \circ \eta_G$, but since $\mathcal{A}$ is a host algebra for G, we have $\pi_{\mathcal{A}} = \gamma_{\mathcal{A}}$ i.e., $\pi \circ q = \gamma \circ q$ and as q is surjective, we get $\pi = \gamma$.

If, in addition, η_G^* is surjective, then every continuous unitary representation π of G/N pulls back to a continuous unitary representation of G which defines a unique representation $\rho_{\mathcal{A}}$ of $\mathcal{A}$ which in turn extends to the representation $\widetilde{\rho}_{\mathcal{A}}$ of $M(\mathcal{A})$ satisfying $\widetilde{\rho}_{\mathcal{A}} \circ \eta_G = \pi \circ q_G$. Further, $\widetilde{I}_N \subseteq \ker \widetilde{\rho}_{\mathcal{A}}$ implies $I_N \subseteq \ker \rho_{\mathcal{A}}$, so that $\widetilde{\rho}_{\mathcal{A}}$ factors via $M(q)\colon M(\mathcal{A}) \to M(\mathcal{B})$ through a strictly continuous representation $\widetilde{\pi}_{\mathcal{B}}$ of $M(\mathcal{B})$, satisfying $\widetilde{\pi}_{\mathcal{B}} \circ \eta_{G/N} = \pi$. This implies that $\eta_{G/N}^*$ is also surjective. ■

Corollary A.7. *Let G_1, G_2 be topological groups and $G := G_1 \times G_2$. If G has a full host algebra $(\mathcal{A}, \eta)$, then G_1 and G_2 have full host algebras $(\mathcal{A}_1, \eta_1)$ and $(\mathcal{A}_2, \eta_2)$ with $\mathcal{A} \cong \mathcal{A}_1 \otimes \mathcal{A}_2$.*

Proof. The existence of host algebras of $G_1 \cong G/(\{\mathbf{1}\} \times G_2)$ and $G_2 \cong G/(G_1 \times \{\mathbf{1}\})$ follows directly from the last statement in Proposition A.6. Now Lemma A.4 applies. ■

Symplectic space

Lemma A.8. *In each countably dimensional symplectic vector space (S, B), there exists a basis $(p_n, q_n)_{n\in\mathbb{N}}$ with*

$$B(p_n, q_m) = \delta_{nm} \quad \textit{and} \quad B(p_n, p_m) = B(q_n, q_m) = 0 \quad \textit{for} \quad n, m \in \mathbb{N}.$$

Then $Ip_n := q_n$ and $Iq_n = -p_n$ defines a complex structure on S for which $(v, w) := B(Iv, w)$ is positive definite and hence defines a (sesquilinear) inner product on S by

$$\langle v, w\rangle := (v, w) + iB(v, w)\,.$$

Moreover $\{q_n \mid n \in \mathbb{N}\}$ is a complex orthonormal basis of S w.r.t. $\langle\cdot,\cdot\rangle$.

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be a linear basis of S. We construct the basis elements p_n, q_n inductively as follows. If $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ are already chosen, pick a minimal m with $e_m \notin \operatorname{span}\{p_1, \ldots, p_k, q_1, \ldots, q_k\}$ and put

$$p_{k+1} := e_m - \sum_{i=1}^{k} \big(B(e_m, q_i)p_i + B(p_i, e_m)q_i\big)$$

to ensure that this element is B-orthogonal to all previous ones. Then pick ℓ minimal, such that $B(p_{k+1}, e_\ell) \neq 0$, put

$$\widetilde{q}_{k+1} := e_\ell - \sum_{i=1}^{k} \big(B(e_\ell, q_i)p_i + B(p_i, e_\ell)q_i\big)$$

and pick $q_{k+1} \in \mathbb{R}\widetilde{q}_{k+1}$ with $B(p_{k+1}, q_{k+1}) = 1$. This process can be repeated ad infinitum and produces the required bases of S because for each k, the span of $p_1, \ldots, p_k, q_1, \ldots, q_k$ contains at least $e_1, \ldots, e_k$.

That $\{q_n \mid n \in \mathbb{N}\}$ a complex orthonormal basis w.r.t. $\langle \cdot, \cdot \rangle$ follows from the definitions.■

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